

CONSUMER SEARCH THEORY

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P.B. Morgan

University of Canterbury,
Christchurch, New Zealand

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ABSTRACT

This thesis considers several aspects of consumer search theory. The thesis begins by giving a history of developments in consumer search theory since its inception in 1961. Various failings of the literature are discussed and the implications of some of these deficiencies are established. In particular, it is shown that fixed demand restrictions can be incorporated into neo-classical search models only with difficulty.

In Chapter 2 an examination of some earlier sequential search models is made. It is found that the restrictions placed on these models are logically inconsistent, except in circumstances which are particularly unusual. A sequential consumer search model, which is not limited by these restrictions, is constructed and compared to the earlier models criticised.

Chapter 3 points out that the search literature has largely ignored the problem of a searcher's rational choice of a preferred set of sellers to sample. The effects of marginal financial search costs, transaction costs and information on sellers' behaviour on this choice are described. Some implications of the analysis for advertising strategies are presented. In addition it is shown that, in the circumstances considered, rational choice of search path results in the optimal sequential search strategy being myopic.

In Chapter 4 a variety of statistical results are offered. The ex ante probability mass function of sequential search lengths is derived and some comparative statics results established. The ex ante probability density function of minimum observed prices resulting from optimal myopic sequential search is then derived, and its

properties are compared to properties which it has previously been inferred to possess. It is found that some of these inferences are not generally valid.

The Stigleresque type of consumer search model is substantially extended in Chapter 5 through the use of the indirect utility function. It is shown that the optimal search strategy cannot exclude the consumer's allocation decision, as previous analyses have done, and thereby relaxes Stigler's fixed order quantity restriction. Optimal search paths for Stigleresque searchers are described. Comparative statics results are determined which describe the dependence of the quantity of search undertaken on the consumer's wealth, psychic search costs and the nature of the searched-for commodity.

In Chapter 6 a generalised search theory is described. This search behaviour is sufficiently general to include both the sequential and Stigleresque search behaviours as special cases. Consequently, the analysis of this chapter resolves the debate over the relative merits of sequential and Stigleresque search rules.

Chapter 7 offers some concluding remarks.

CHAPTER I

AN INTRODUCTION TO SEARCH PROBLEMS

SECTION 1-1: SEARCH AS AN ECONOMIC ACTIVITY: THE SEMINAL CONCEPTS

'Price' is a term used with great facility and frequency by economists of all kinds. Much effort is directed towards understanding the effects of price changes on economic variables but most of these arguments and theories make severe assumptions about prices; assumptions which inhibit insights into the establishment, movements and persistence of price structures easily observable in the real world. One characteristic of real world price structures which is of sufficient importance to prompt all of us to window shop, take notice of advertisements and take trouble to amass some considerable knowledge of price structures affecting us, is price dispersion. Yet the existence of price dispersion is flatly denied by main-line economic theory. For gains in analytic simplicity we have, from before this century, created a variety of devices to ensure the complications of price dispersion did not trouble the analysis of economic theories. The most persistently invoked device is the Walrasian auctioneer, an awesome creature mysteriously endowed with the tâtonnement power to elucidate a single market clearing vector of prices to which all traders must adhere. Similar in concept and effect is Edgeworth's device of allowing provisional contracts between buyers and sellers - provisional in that should one agent locate a more advantageous contract he may disregard the first. Only when no advantage can be gained by additional recontracting does trading take place. The assumptions surrounding prices are probably at their most extreme in the general equilibrium models formulated by Arrow and Debreu. In these models, agents are not only overseen by the Walrasian auctioneer but

enjoy the advantages of perfect futures markets as well, enabling a Pareto optimal set of contracts to be decided upon now contingent upon all future possible events. Whatever the name and the mechanism assumed, all these devices have a central feature. No trading takes place until a single market clearing vector of prices is established. Such a restriction greatly facilitates the analysis of the theorists' models and the understanding that results can easily be said to justify the costs of the restriction. Nobody argues that the tâtonnement process exists in real markets. The severity of the necessary supporting assumptions of perfect knowledge, costless information, instantaneous trades etc. is obviously too great for any real world credibility to be attached to the process.

Analytic gains have been made, but as part of the price paid for them we have little insight into how dispersed price structures arise and persist. Only recently have any attempts been made to formulate plausible theories describing price dispersion. The first attempt at explaining why price dispersion is present in markets where commodities are homogeneous, at least with respect to their physical characteristics, came only in 1961 when G.J. Stigler published the seminal paper on search as an information-producing economic activity [50]. The theories on search which have developed are a promising attempt to incorporate price dispersion into economic theory. They are motivated by the simple observation that, in the real world, information about uncertainties is desirable but costly. Stigler, presumably prompted by this observation and musings of the kind above, wrote [50] "Price dispersion is a manifestation - and, indeed, it is the measure - of ignorance in the market" and "One should hardly have to tell academicians that information is a valuable resource: knowledge *is* power. And yet it occupies a slum

dwelling in the town of economics. Mostly it is ignored: the best technology is assumed to be known; the relationship of commodities to consumer preferences is a datum. And one of the information-producing industries, advertising, is treated with a hostility that economists normally reserve for tariffs or monopolists." A second paper [51] in very similar vein to the first is the seminal paper on 'job search'. Separate, but complementary, literatures of increasing sophistication are accumulating behind each of these papers. A review of these literatures, biased towards consumer search, is contained in Section 1-2. In anticipation of this review, Stigler's seminal concepts on search as an information-producing activity are now presented.

The search literature subsequent to Stigler has incompletely captured important aspects of the problem, as this thesis will show. The seminal paper is more conceptual than formally precise. Subsequent criticisms make it appear as though the main thrust of the paper was to describe individual search behaviour but in fact the main thrust was an attempt to explain the causal factors of price dispersion. Stigler's consumer search model [50] initially considers an individual consumer who wishes to purchase a particular quantity of a particular commodity. Prices in this commodity market are dispersed and Stigler introduces the search activity by saying "A buyer (or seller) who wishes to ascertain the most favourable price must canvass various sellers (or buyers) - a phenomenon I shall term 'search'." [50, p.213] The quantity demanded is determined prior to search and is not revised as search proceeds. How the consumer determines his demand is not explained. The commodity is assumed homogeneous throughout the market to avoid considerations of difference in quality. Relative likelihoods of different values of the price are described by a probability density

function (p.d.f.), $f(p)$, the functional form and parameters of which are known exactly to the consumer. With a fixed quantity demanded and no transaction costs, the consumer's most desired price is the lowest since this minimises the fraction of the consumer's wealth allocated to the fixed quantity purchase of the commodity. The 'savings' from search can then be allocated to other commodities. Stigler never explicitly stated this as the motive for search as an economic activity but it is difficult to see what else he could have had in mind. He states "For any buyer the expected savings from an additional unit of search will be approximately the quantity (q) he wishes to purchase times the expected reduction in price as a result of the search" and that "The expected savings from given search will be greater, the greater the dispersion of prices. The saving will also obviously be greater, the greater the expenditure on the commodity" [50, p.215]. The Stigleresque consumer's search is costly and of diminishing marginal value. "Whatever the precise distribution of prices, it is certain that increased search will yield diminishing returns as measured by the expected reduction in the minimum asking price" [50, p.215] and "The cost of search, for a consumer, may be taken as approximately proportional to the number of (identified) sellers approached, for the chief cost is time. This cost need not be equal for all consumers, of course: aside from differences in taste, time will be more valuable to a person with a larger income" [50, p.216]. Correctly assumed, but not proven, by Stigler is that non-zero costs and diminishing marginal expected value of search are sufficient conditions to ensure the Stigleresque searcher will wish to 'canvass' only a finite number of sellers. Stigler gave the rule for determining the amount of search which minimises expected expenditure on the commodity as "If the cost of search is equated to its expected marginal

return, the optimum amount of search will be found" [50, p.216].

A qualification to this rule is "The equation defining optimum search is unambiguous only if a unique purchase is being made - a house, a particular used book, etc. If purchases are repetitive, the volume of purchases based upon the search must be considered" [50, p.218]. The quantity of search calculation is carried out before search commences and the amount of search so decided upon is not revised as search proceeds.

Stigler devoted the latter and larger part of his paper to sellers' behaviour. It is perhaps surprising, therefore, that relatively few of the subsequent writers explicitly consider the sellers' side of the market. Stigler noted that sellers may also indulge in search - "Of course, the sellers can also engage in search and, in the case of unique items, will occasionally do so in the literal fashion that buyers do. In this - empirically unimportant - case, the optimum amount of search will be such that the marginal cost of search equals the expected increase in receipts, strictly parallel to the analysis for buyers" [50, p.216]. For the most part, however, the Stigleresque seller is described as "Each dealer sets a selling price, p , and makes sales to all buyers for whom this is the minimum asking price", and "We should generally expect the high-price sellers to be small volume sellers" [50, p.217]. The Stigleresque seller is thus restricted to offering the same price to each buyer who contacts him.

As indicated earlier, a desirable outcome from the pursuit of search theories is a theory explaining persistent price dispersion in a market at equilibrium, although a new definition of 'equilibrium' may have to be coined for this occasion. Stigler asserts persistent price dispersion to be the consequence of a perpetual stock of ignorance on

the part of both buyers and sellers. He says, "The maintenance of appreciable dispersion of prices arises chiefly out of the fact that knowledge becomes obsolete. The conditions of supply and demand, and therefore the distribution of asking prices, change over time. There is no method by which buyers and sellers can ascertain the new average price in the market appropriate to the new conditions except by search. Sellers cannot maintain perfect correlation of successive prices, even if they wish to do so, because of the costs of search. Buyers accordingly cannot make the amount of investment in search that perfect correlation of prices would justify. The greater the instability of supply and/or demand conditions, therefore, the greater the dispersion of prices will be" [50, p.220].

Theories of search thus rely neither upon a Walrasian auctioneer nor upon the restriction that all arbitrage occur at the same instant of time. Removing these simplifications, which are crucial to main-line economic theory, poses severe problems. The most immediate problem is to define an equilibrium in the resulting markets. The Arrow-Debreu equilibrium notion that net demand must equal total resources at equilibrium (see [8, p.76]) is no longer adequate. The real world phenomena of unsold stocks and unsatisfied demands may well be present in the new equilibrium (see [18, p.261] for one example of an equilibrium with this property). A description of an equilibrium is meaningful only if the equilibrium exists. The problems of proving existence and examining if such equilibria are attainable are set in the context of dynamic markets in which different agents may trade at different times and at different real prices. Consequently, these tasks will be considerably more difficult than the comparable proofs evolved for the perfectly competitive model.

Stigler continued in his paper to discuss the role of advertising but we shall now briefly turn our attention to his seminal paper on job search. The Stigleresque job search model is a conceptual mirror image of the consumer search model. The principal distinction is that it is now the highest wage sought by a searching worker instead of the lowest commodity price. A closely similar search rule was asserted - "A worker will search for wage offers (and an employer will search for wage demands) until the expected marginal return equals the marginal cost of search" [51, p.96]. Again the amount of search carried out is determined prior to search and is not revised as search proceeds. It would be repetitious to outline the job search model fully. Accordingly, the reader is invited to infer the structure of the job search model by analogy to the consumer search model described above.

The seminal concepts have been described. In the following section an account is given of how subsequent authors have developed and criticised Stigler's ideas. In Section 1-3, the reader's attention is drawn to inadequacies still remaining in search models and a description of how this thesis resolves some of them. Section 1-4 describes some of the possible variations on the search problem theme, explains how the thesis elucidates their differences, compares their conclusions, and extends them.

Section 1-2: A HISTORY OF DEVELOPMENTS IN SEARCH MODELS

This section presents the literatures that have accumulated behind Stigler's seminal papers [50] and [51], and criticisms and refinements of the seminal concepts. Although larger and more varied, the job search literature is given less attention than the consumer search literature, since this thesis is primarily concerned with consumer search.

The early papers of either literature are concerned with refining the individual searcher's optimal search rule. McCall was the first researcher (1965) to draw attention to the distinction between a Stigleresque fixed-sample-size search rule and a sequential search rule [31]. In a later paper (1970) McCall [32] specifically applied a sequential search rule to a job search problem and noted that Bayesian pre-posterior analysis is applicable to those search problems where the p.d.f. of wages (prices) is in some way not completely known to the searcher. McCall's survey paper [30] repeated these ideas and in addition called attention to the use of martingales in search problems. These three initial modifications to Stigler's original analyses are commonly included in subsequent research and are elaborated upon in Sections 2-3, 2-5, 2-6, 2-7, 3-7 and Chapter 6. McCall's job search comments gave impetus to similar work on consumer search. First Nelson [35] and Telser [52] and then Gastwirth [11] and Rothschild [41], [43] criticised Stigler's consumer search rule. Their criticism was that sequential search rules are always optimal and that "fixed-sample-size rules ... are not the best search procedures and are in some circumstances simply silly" [43, p.691]. The Telser, Gastwirth, Rothschild consumer search models are similar to the Stigler model, but differ in that marginal search costs are solely financial and the same for each observation, the p.d.f. of selling prices $f(p)$ is not always completely known, and, as mentioned, the search rules are sequential. Both the Telser and Gastwirth papers [52], [11] are Monte Carlo studies directed at determining the sensitivity of search rules to the parameters conditioning them. Telser's results, derived for a variety of p.d.f.'s $f(p)$, first completely known and secondly completely unknown, indicate "the gain from search is modest unless the (*financial*)* marginal cost

* My italics

of search is a very small percentage of the minimum price or the range (*of prices*)* is very large" [52, p.45]. Gastwirth's study [11] supports Stigler's assertion that positive correlation between price observations reduces the optimal search length and shows that misspecifying the optimal search rule through incorrect knowledge of the p.d.f. of selling prices $f(p)$ can result in expensive errors in the number of observations taken. However, Gastwirth's consumer is not allowed to utilise his price observations to check if such a misspecification has occurred. It would be interesting to see if some such information feedback system significantly reduced the expense of such errors. Rothschild [43] adopts a more formal and rigorous approach to consumer search with his model. His consumer is provided with incomplete prior information about a discrete probability mass function (p.m.f.) of selling prices $f(p)$ and utilises Bayes rule to refine this information with price observations taken during search. Rothschild examines the optimal search rule for his consumer and concludes that "the qualitative behaviour of persons searching optimally from unknown distributions is the same as that of persons searching optimally from known distributions" [43, p.708].

A much more general approach to sequential search problems is taken by Kohn and Shavell [20]. Their analysis is conducted specifically in utility terms, breaking with the use of consumer expenditure minimisation or job income stream maximisation as a proxy for utility maximisation. It will be shown in Section 2-6 that using consumer expenditure minimisation as a proxy for utility maximisation is valid only when the consumer's utility function belongs to a particularly heavily restricted class of utility functions. These necessary restrictions have been ignored entirely by previous authors. Kohn and Shavell

* My italics

consider a searcher who may take observations x_t sequentially from a population X over which some probability distribution for x_t exists. This probability distribution may be completely or partly known to the searcher. Upon making the j th observation x_j the searcher "enjoys an immediate utility payoff of $k_j(x_j)$ " [20, p.95]. In the case of consumer search this is interpreted as the marginal search cost of the j th observation. Kohn and Shavell show that "in very general circumstances the optimal decision rule of an expected utility maximiser takes the form of a switchpoint level of utility s . If the utility of the best x_t available so far is higher than s , the search will end; otherwise it will continue" [20, p.93]. A variety of comparative statics results about the switchpoint s are presented, describing its behaviour w.r.t. changes in time preference, search costs, risk attitudes and the probability distribution over the population X . Proof is provided that optimal sequential stopping rules exist either when utility is bounded above or when there is Bayesian learning about the probability distribution over X and independence between observations x_t .

Authors such as Danforth [6], [7], Pissarides [37], Whipple [55], Lippman and McCall [23], to name a few, have substantially refined models of individuals' job search behaviour from McCall's original sequential search model [32]. Whipple's model [55], for example, incorporates non-linear utility functions, endogenously changing skill levels and preferences for different jobs, and enlarges the search decision variables from merely the acceptance wage to the rates of saving and search as well. Pissarides [37] also utilises non-linear utility functions in the development of a job searcher's optimal sequential search rule. Comparative statics and dynamic properties of the rule are examined for changes in the p.d.f. of wages, search costs

and it is shown that under some conditions it will be optimal for search to be temporarily halted and later resumed. In Section 6-1 similar consumer behaviour is described as part of a 'generalised consumer search' model. The generality of this model is sufficiently great for it to be able to describe both fixed-sample-size search rules of the Stigler type and sequential search rules of the Rothschild, Gastwirth etc. type. Lippman and McCall [23] have produced a dynamic model in which they establish the optimal job search rule when the labour market's wage distribution varies from period to period according to a known Markov process. Lippman and McCall are also the authors of the currently most comprehensive survey [24], [25] of the job search literature.

Models attempting to explain how price dispersion may persist in a market at equilibrium must consider not only consumers' search behaviour but also consider sellers' responses to consumers' behaviour. The models described so far are one-sided in that they preserve Stigler's original assumption of passivity on the part of sellers. Models adopting the opposite stance have been developed by a number of authors. In these models, consumer behaviour is either passive or else unspecified. Typically a non-passive seller is considered not to indulge in search activity in the manner of a consumer because he finds it excessively costly. Instead, the seller is considered to wait until contacted by a searching consumer to whom the seller offers a price which the consumer may either accept or reject. The non-passivity is taken to mean that the seller may offer different prices to different buyers or may alter his price so as to alter the rate at which buyers contact him. A description of the first type of non-passive seller behaviour is provided by Manning [28]. Manning's seller ignores the actions of other

sellers and offers the k th searcher who contacts him the k th price in a sequence of selling prices. The values of the prices in the sequence depend upon the seller's reservation price, the number of buyers who contact him and the p.d.f. of reservation prices the seller assumes buyers possess. The sequence is not revised as buyers contact the seller. If all buyers are identical the sequence of prices offered is strictly decreasing. Fisher [10] provides a model in which competing sellers may offer different prices to buyers. The buyers' search behaviour is directed by sellers' relative prices to the extent that a seller offering a particular price must receive at least as many enquiries from searching buyers as a seller offering a higher price. Apart from this, buyers' search behaviour is left unspecified. The model is, however, relatively uninteresting in that sellers have identical cost curves, consumers have identical demand functions and search is costless. Not surprisingly, the model shows that any initial price dispersion eventually collapses to the perfectly competitive price. A more sophisticated version of Fisher's model has been put forward by Hey [15]. Hey's searchers use a Stigleresque fixed-sample-size search rule and buy an endogenously determined quantity from the seller offering the smallest observed price. Sellers use buyers' reactions to their price offers to refine their knowledge of the buyers' identical demand functions and attempt to maximise profits. In contrast to Fisher's model, Hey concludes that any initial price dispersion collapses to the monopolistic price rather than the perfectly competitive price. However, Hey's results are invalid if his buyers are permitted to search sequentially. A comment by Hey [15, p.486, footnote 8] that sequential search is not applicable to his model is incorrect and invalidly referenced. Rothschild has presented a model [42] in which

the efforts of sellers to maximise expected discounted future profits results in persistent price dispersion. The sellers possess the same initial market opportunities and the same costs but different experiences with customer acceptances and refusals of offered selling prices lead sellers to eventually offer different prices for all time. As Rothschild comments, his sellers are not truly competitive since "The model assumed that the flow of customers was independent of the stores' actions" [42, p.200]. Further comment on some of these models is contained in a survey article by Rothschild [41].

A more aggressive seller is envisaged by Butters [3] who considers advertising strategies adopted by sellers competing for the favours of potential buyers. The advertisements are mailed at random to potential buyers and contain a price offer. The model is a type of Stigleresque sellers model in that the sellers must decide upon the number of advertisements to mail at the beginning of a period and cannot revise the decision. A potential buyer must either accept the lowest price offer received within the period or else leave the market. An effort by Butters to incorporate search by buyers into the model is very rudimentary and, for the most part, buyers are taken to be passive. The p.d.f. of transaction prices for the model is derived and its properties examined.

An empirical effort to examine the properties of the probability distributions of search lengths and transaction prices has been made by Axell [1]. His Monte Carlo study is based upon a sequential search model in which sellers are totally passive and buyers revise the knowledge of a partly unknown p.d.f. of selling prices as their searches proceed. Various empirical comparative statics results which Axell infers from his numerical results are compared to the theoretical results derived

in Sections 4-3 and 4-4.

To model price dispersion realistically, it is necessary to combine optimal search by buyers with optimal seller response to such search. Such a combined analysis must be dynamic in some sense because search is necessarily a time consuming activity. Efforts to construct such models have been made by Mortensen [33], Lucas and Prescott [27], Ioannides [18] and Telser [53]. Mortensen's analysis was the first to appear (1970) and is an explicitly dynamic analysis of a labour market in which both workers and employers display competitive behaviour. Workers may sequentially search out employment opportunities in decentralised labour markets. Employers are competitive in their demand for labour in that the flow of searching workers to an employer is partly determined by the wage offered by the employer. Given an exogenously fixed turnover rate in the employed labour force, the model endogenously determines the unemployment rate and the rate of wage inflation. Many comparative statics results are also provided but few of these concern changes in the p.d.f. of wages. Nevertheless, Mortensen's model remains significant as the first effort to place the economics of information from search in a dynamic context. With this in mind it is interesting to compare it to Lucas and Prescott's model of wage dispersion in equilibrium labour markets. Theirs is a model of perfectly clearing sub-markets isolated from each other in some way. Workers may move between sub-markets but only at the cost of one period's unemployment. Product demands in the sub-markets are subject to random shocks which give rise to wage differentials between sub-markets which, in turn, provide the incentive for workers to move between sub-markets. Wage dispersion and positive unemployment are shown to persist and some loosely argued welfare results are presented. However, as Butters [3]

comments, the model contributes little towards understanding the economics of information. Lucas and Prescott's worker is endowed with complete knowledge of his sub-market's product demand, workforce, and the p.d.f.'s of all present and future states of all sub-markets. Such a plethora of information gives the worker little opportunity to reduce the uncertainty he faces by searching. To date, the most refined model of nontâtonnement adjustment of price dispersion is Ioannides' model [18]. Buyers and sellers in a market may contact each other randomly. Should a contact occur, the seller will offer the buyer a price which reflects the seller's assessment of buyers' reservation prices. The buyer either accepts or rejects the offered price, without revealing his reservation price. A single unit of the commodity is exchanged should buyer and seller agree on a price. Given both buyers and sellers are expected utility maximisers, both the demand and supply sides of the market are solved simultaneously for an equilibrium defined as stationarity in the number of market participants and the p.d.f.'s of asking and reservation prices. Within such an equilibrium, prices are dispersed and some market participants' supplies and demands are not satisfied. Such equilibria are shown to exist provided market participants enter the market in a particular manner and remain in the market for only a finite length of time. Ioannides does not show how such an equilibrium is attained.

This completes the presentation of what might be called "main stream" search literature in that it follows directly from Stigler's seminal concepts. There are also a number of papers impinging on to the main stream and some of these are now presented. Stigler's original assumption of a homogeneous commodity market is retained in all of the models presented so far, preventing any of them shedding any light on

the allied problem of demand for information about product qualities. Kihlstrom [19] has constructed a Bayesian model of consumer demand for such information in which he adopts the Lancaster notion (see [22, p.113-118]) that consumers purchase commodities not for the substance of the commodities, but for the utility provided by "attributes" possessed by the commodities. A commodity's quality is measured by the amounts of these attributes it possesses. In Kihlstrom's model consumers may purchase observations on variables related to commodity qualities prior to purchasing quantities of the commodities. The consumers' demands for the commodities are thus dependent on the consumer's observations on commodity qualities.

Also common to all the search models presented is the underlying assumption that buyers' preferences are fixed. This is an assumption standard in neo-classical theory and one that is trivially valid for any static analysis. Search, however, is a dynamic activity so the question of how a searcher may revise his preferences is a meaningful problem. Revision of preferences was recognised as a problem in dynamic problems as long ago as 1898 by T. Veblen [54] and a large literature on the problem now exists. Recent general treatments of the problem have been presented by Gorman [12] and Krelle [21]. The problem has been extended by Cyert and De Groot [5] and by Long and Manning [26] to include a consumer's learning about his preferences. Such a consumer selects a present consumption pattern to maximise the utility derived from present and future consumption when this depends upon what he learns about his preferences from the consumption pattern chosen presently.

Many papers have been presented which examine problems involving learning under conditions of uncertainty. Only those directly concerned

with or closely related to the problems raised by Stigler are presented here.

That the search literature contains significant deficiencies is not surprising considering the youth of this branch of economic theory. The purpose of the next section is to point out the principal deficiencies.

SECTION 1-3: DEFICIENCIES IN PRESENT SEARCH MODELS

This section presents aspects of the search activity in which present consumer search models are usually deficient. Some of these deficiencies are at least partly remedied in later sections of this thesis.

In standard neo-classical analysis, an economic agent is restricted in his actions by market conditions which are the result of other agents' actions. That a considerable number of additional restrictions are placed upon agents' actions in search models indicates the presence and nature of significant deficiencies in these models. Most of the restrictions can be placed into one of two categories - restrictions on the consumer's utility function and restrictions on the degree of uncertainty in the market. Given there are many restrictions in use, it is immediate that there is no such single thing as the 'search problem'. A large variety of search problems is generated by any consistent combination of these restrictions. It seems that, by and large, writers on search problems have not clearly realised that this variety exists for few clearly interpret the problems they analyse. That one writer's results have provoked argument with another's is sometimes a consequence of different models being analysed and not a consequence of incorrect analysis by either party.

This is the heart of the conflict over which of the Stigleresque fixed-sample-size search rule and the sequential search rule of McCall, Rothschild etc. is optimal. The conflict is resolved in Chapter 6 where a theoretical justification is given for the cold fact that both Stigleresque and sequential search behaviour are often used in the real world.

Some of the variations possible on the search problem theme are discussed in the next section. The restrictions giving rise to these variations are now listed.

Restrictions on the Consumer's Utility Function:

(i) Frequently (eg. [1], [3], [11], [43], [50], [52], [53]) the consumer's utility function is restricted to those functions for which the consumer will always demand one unit of the searched for commodity, whatever the price-income situation facing the consumer. That demand is one unit in particular is merely a matter of choice of units. That demand for this commodity is independent of consumer wealth and the price of the commodity is shown in Section 1-4 to be severely restrictive in a neo-classical world. It is intuitive that the consumer's budget allocation decision must be an integral part of his adaptive search strategy. Observed prices influence his expectations on the utility maximising allocation possible from extending his search further and so play an important part in determining the extent of his search and the gains received from it. It is therefore surprising that, despite the considerable attention paid to search rules, no writers (with one possible exception) have allowed that prices observed during search and changes in consumer wealth due to financial costs of search will usually cause changes in the consumer's demands. This is one of

the two greatest single deficiencies in the consumer search literature. Kohn and Shaveil [20] can possibly be excepted from this criticism since they approach sequential search problems in a sufficiently general manner to allow an interpretation of their work as a consumer following an adaptive allocation and search rule. However, they do not discuss the possibility.

(ii) Frequently it is assumed that a consumer's utility function is such that he will always choose to accompany his exit from the market with the purchase of a non-zero quantity of the searched for commodity. The consumer's market-exit decisions are thus excluded from discussion (see [1], [3], [11], [43], [50], [52]). If restriction (i) is present in a search model then it is necessary that restriction (ii) be present also. Non-entry to the market or exit from the market without purchasing any of commodity 1 is equivalent to demanding zero units of the searched for commodity and this would violate restriction (i).

(iii) Almost invariably allied to restrictions (i) and (ii) is the restriction that all consumers choose to search. There is no consideration of a consumer decision on market entry (see [1], [3], [11], [15], [18], [32], [35], [43], [50], [52]).

(iv) Many analyses are carried out in monetary terms rather than utility terms. The underlying restriction in these analyses is that the consumer's utility function is such that minimisation of expenditure on the searched for commodity is equivalent to utility maximisation (see [1], [3], [11], [43], [50], [52]).

(v) Search costs are frequently assumed to be solely financial ie. to have no psychic component. They are therefore disincentives for search only in that they reduce the utility attainable by reducing the

consumer's initial wealth (see [1], [3], [11], [43], [52]).

(vi) All search theorists (Kohn and Shavell [20] may again possibly be excepted) assume their searchers' utility functions are such that they will not choose to change from potential buyer to potential seller (or vice versa) during their searches. In reality, however, a potential buyer of additional units of the searched for commodity could observe prices high enough to persuade him to sell some of his endowment of the commodity and allocate the proceeds to purchases of other commodities. For instance, a person seeking to hire a plumber may find the prices high enough to induce him to give up some of his leisure time to perform plumbing services for others. The restriction also avoids having to consider speculative activities, always a possibility in real world markets with dispersed prices and incomplete individual knowledge of these prices. A commodity speculator must both buy and sell the commodity. Assuming no individual can indulge in both buying and selling therefore has the effect of assuming speculation does not exist.

Restrictions on the Level of Uncertainty in the Market:

(vii) Searchers are assumed to have either an exact knowledge of their wealth at all times or to have some way of determining their utility maximising allocation of wealth without an exact knowledge of what their wealth actually is. This assumption is never explicitly stated but is implicit in every search model known to this writer.

(viii) Searchers are assumed to have an exact knowledge of all search costs possible at any time before or during search. The assumption avoids the complications involved in a consideration of the searcher's beliefs about search costs - a consideration which would not contribute

significantly to the understandings gained from analysing search models.

(ix) The searched for commodity is assumed to be homogeneous throughout its market. Yet this assumption denies one of the sources of price dispersion, the phenomenon which search models seek to explain. The assumption is made to avoid what is essentially an aggregation problem. It is obvious that a rational consumer will choose to buy a Super Suction vacuum cleaner for \$50 from a seller who offers a three year warrantee rather than for \$50 from a seller who offers no warrantee at all. The question is, therefore, should these two vacuum cleaners be considered identical commodities? To what degree of coarseness should any aggregation be carried? A possible escape from this problem is to adopt the Lancasterian notion that a consumer purchases a commodity for the sake of 'attributes' possessed by the commodity (see [22, p.113-118]). This concept sidesteps the need to consider differences in commodities' physical characteristics, warranties, etc. Olsen [36] adopts such an attitude towards a housing market where it is assumed that an attribute called "housing service" is bought and sold. However, commodity heterogeneity and the Lancasterian analogue of restriction (i), that the consumer's demand for the searched for attribute is always one unit, together imply that different quantities of commodities with this attribute will be purchased. Variable quantities demanded are a degree of freedom earlier authors have found analytically difficult and consequently they have chosen to assume homogeneity. Search models with discretely and continuously variable demands are analysed in Chapters 2 and 5 respectively.

(x) Transaction costs are assumed not to exist. The consequences of including transaction costs in search models are set out in Chapters 2, 3 and 6.

(xi) If a searcher is to contact a certain set of sellers, then he will usually have a most preferred sequence in which to do this. Yet search models assume such sequences (called 'search paths' from now on) are indistinguishable in terms of costs and benefits. Search paths are examined in Chapter 3.

(xii) The p.d.f. of selling prices is assumed static throughout the whole of an individual's search. This restriction is very common because of the great gains it offers in statistical simplicity. It implies both some kind of passivity on the part of sellers and some restrictions on their supply functions. There is no doubt that changes in the p.d.f. of selling prices do occur. Moreover, individuals are likely to perceive these changes only slowly so these changes must be a contributing factor to maintaining ignorance about prices and price dispersion in the market. The analytic difficulties encountered without this restriction are the principal reason why no researcher has yet succeeded in constructing a model showing a path by which a dis-equilibrium market with dispersed prices may attain an equilibrium with dispersed prices.

(xiii) Sellers are assumed to offer a selling price to every buyer who contacts them and to be able to honour this offer should a buyer later return to take up an offer. This implies a serious restriction on both the sellers' supply functions and the amount of information about the market which must be possessed by individual sellers, each of whom must be able to satisfy any total demand made upon him by buyers who have previously contacted him.

The usual motivation for introducing these restrictions into consumer search models seems to be a desire for analytical simplicity. Some of the restrictions can be introduced without greatly altering the nature of the model. For example, introducing restriction (viii) merely removes the need for using expected instead of actual costs. Most of these restrictions, however, have a substantial effect on the nature of the model if introduced. Some of these effects are not obvious. For example, in the next section it is shown that restrictions (i) and (v) can be combined consistently if and only if the consumer's preferences are not smooth. To the extent that all the restrictions damp factors giving rise to price dispersion, they all inhibit Stigler's original aim in formulating the search problem - seeking the causes of persistent price dispersion.

The exclusion of the consumer allocation decision from the search rule was labelled earlier as one of the two greatest deficiencies in the consumer search literature. The second of these deficiencies is the incomplete fashion in which the literature specifies the way in which time constrains search. Search is necessarily a dynamic activity and time is therefore an important consideration for all searchers. For example, trading is commonly permitted only in certain periods - retailers are commonly closed from evening until the next morning. Search is therefore confined to the daytime. The essential point is that the search method which is optimal for a searcher depends upon the constraints placed upon the searcher by time and other factors. Consider the following three examples.

Example 1: A person wishes to build a house. House building ceases with the advent of winter in two months time. It will be far more costly to build the house next spring. One month is the time required

to build a house and builders need one month to prepare quotations for the job. Consequently, the person must solicit quotations now and may make more than one observation on prices by asking more than one builder for a quotation. Search thus lasts one period (one month) and the searcher takes at least one observation, possibly more, within the period. Time constraints make Stigleresque search the optimal method.

Example 2: A person wishes to buy an object for which price quotations can be obtained only by visiting the seller in person. The searcher's transport difficulties prevent him from visiting more than one seller on any one day (a period). The person's search may be carried out over many days but on any one day only a single price observation is made. The constraints restricting the searcher make sequential search optimal.

Example 3: A person wishing to buy an object is able to read advertisements from sellers every morning in his daily paper. These advertisements carry selling price offers and thus constitute price observations. The number of advertisements in the paper may vary from day to day. If none of the offered prices on one morning are satisfactory, the searcher may wait until the next day to see what is offered then. Search may therefore persist over many periods and the number of observations taken in different periods may differ. Such search behaviour will be called "generalised search". Generalised search is an approach to search which is more capable than previous efforts of describing the search phenomena observed in the real world. It explicitly considers the consumer's market entry decision, his learning about the market he faces, the adaptive manner in which he decides upon the number of periods over which to persist in his search and the number of observations to take in successive periods. The approach specifically

includes the consumer's allocation problem allowing his demand to be variable as search proceeds and allowing a general description of how he selects a seller (if any) from whom he purchases the chosen quantity demanded.

Some form of adaptive search is always optimal. The above examples demonstrate that in some circumstances the application of the optimal adaptive rule may result in behaviour identical to that prescribed by the Stigleresque search rule or by the sequential search rule. Indeed, both behaviours are frequently observed in the real world. Sequential search behaviour is examined in Chapters 2, 3 and 4, Stigleresque search is examined in Chapter 5 and generalised search in Chapter 6.

An examination of the search literature from its conception in 1961 to the present soon shows that the research effort directed towards search and the economics of information is steadily increasing. Increasing with it is the need for a rigorous framework within which research efforts can be compared formally and clearly. One such framework, wherein models are characterised in terms of the restrictions listed above, is set out in the next section.

SECTION 1-4: NOTATION AND VARIATIONS OF THE 'SEARCH PROBLEM'

The formal analysis presented in this thesis begins in this chapter. It is therefore necessary to present some of the notation and assumptions used in the thesis. Not all of the assumptions are invoked in all sections of the thesis. A note of where assumptions are invoked is given alongside the statements of the assumptions. Additional notation is introduced as required in subsequent chapters to avoid burdening the reader with notation not immediately in use. All

notation used is collected in Appendix 1 for the reader's convenience. The notation used follows symbols in standard use in the economics literature as much as possible to facilitate comparison of this thesis to other works.

The thesis views the search problem against the background of the neo-classical exchange model. In the standard neo-classical model the consumer faces known and fixed prices for all commodities. Given these prices his problem is to achieve his highest attainable level of utility subject to his wealth constraint.

$\omega = (\omega_1, \omega_2, \dots, \omega_\ell)$ - the vector of the consumer's initial endowments of commodities 1 to ℓ .

$p = (p_1, p_2, \dots, p_\ell)$ - the vector of market prices for commodities 1 to ℓ .

The following assumption is made to assist comparison of the analysis carried out in this thesis to the standard neo-classical analysis.

Assumption: The markets for commodities 2 to ℓ are perfectly competitive.

1-4-1

This assumption relegates the consumer to the role of price taker in these markets and requires that the prices of commodities 2 to ℓ are not dispersed and are known to the consumer.

Assumption (1-4-1) is retained throughout the thesis.

$U(x_1, x_2, \dots, x_\ell)$ - the consumer's direct utility function. U summarises the consumer's preferences over his set of feasible consumptions

$$x = \{(x_1, x_2, \dots, x_\ell)\},$$

M - the consumer's wealth prior to search. In the neo-classical model of a private ownership economy M is a function of prices p and initial allocations ω .

$$M = p_1\omega_1 + p_2\omega_2 + \dots + p_\ell\omega_\ell + \text{profits}$$

1-4-2

For a complete specification of profits in a private ownership economy

the reader may consult Debreu [8, p.78]. The standard neo-classical consumer problem is therefore to

$$\max_{x_1, \dots, x_\ell} U(x_1, \dots, x_\ell)$$

$$\text{subject to} \quad p_1 x_1 + \dots + p_\ell x_\ell \leq M \quad 1-4-3$$

$x_i^* = x_i(p_1, p_2, \dots, p_\ell, M)$ - the consumer's demand for commodity i ,

$i = 1, \dots, \ell$. x_1^*, \dots, x_ℓ^* are the solutions to (1-4-3) and in general are correspondences of all prices and wealth. For the purposes of this thesis, however, the x_i^* will be assumed to be functions. The properties of the demand functions x_i^* depend upon the properties of the utility function U . Restrictions placed upon demand functions therefore imply restrictions placed upon utility functions. This is discussed later in this section. For now, note that the x_i^* are continuous functions if U is continuous and strictly quasi-concave w.r.t. x_1, \dots, x_ℓ .

Assumption: $x_1^* \geq 0$. 1-4-4

(1-4-4) is a restriction of x_1^* , and therefore of U , that is invoked throughout the whole of the thesis. If $x_1^* < 0$, commodity 1 would be some type of consumer output, such as labour. (1-4-4) thus restricts the attention of the thesis to consumer search and prevents any consideration of job search. (1-4-4) is equivalent to applying restriction (vi) of Section 1-3 and so precludes discussion of the phenomena mentioned therein.

The price dispersion considered in search problems makes a search theorist's consumer a far less advantaged creature than a standard neo-classical consumer. A glance at (1-4-2) quickly shows that any uncertainty in any of p_1, \dots, p_ℓ means that the consumer's wealth M is also uncertain. This makes the solution to the consumer problem (1-4-3) much more difficult. Assumption (1-4-1) restricts price dispersion to

only the market for commodity 1 but this is sufficient to make M uncertain if the consumer has a non-zero initial endowment of commodity 1 ($\omega_1 \neq 0$). In search problems the value of p_1 is uncertain both before and during search. To avoid the difficulties associated with M being uncertain it is assumed throughout the thesis that the consumer has a zero initial allocation of commodity 1.

Assumption: $\omega_1 = 0$.

1-4-5

However, (1-4-5) by itself is not sufficient to ensure that M is known with certainty. Even if $\omega_1 = 0$, M will be uncertain if the consumer speculates by both buying and selling commodity 1 as he searches. Consequently it is always assumed that the consumer does not speculate in commodity 1.

Assumption: There is no speculation in commodity 1.

1-4-6

$I(p_1, \dots, p_\ell, M)$ - the consumer's indirect utility function. The

indirect utility function is the objective function of the dual problem to the budget constrained direct utility maximisation problem.

I is derived from the direct utility function U by replacing the arguments x_i of the direct utility function by the demand functions $x_i(p_1, \dots, p_\ell, M)$, ie.

$$I(p_1, \dots, p_\ell, M) =$$

$$U(x_1(p_1, \dots, p_\ell, M), x_2(p_1, \dots, p_\ell, M), \dots, x_\ell(p_1, \dots, p_\ell, M)) \quad 1-4-7$$

More complete explanations of the derivation and properties of the indirect utility function are contained in [13, p.197-208], [16], [17] and [49]. The consumer search problems are concerned with changes in consumer welfare resulting from changes in prices and wealth. These variables are arguments of the indirect utility function while only appearing in the constraint of the direct utility maximisation problem. Consequently, the indirect utility function is more immediately applicable

to the analysis of search problems than is the direct utility function. Kohn and Shavell [20] recognise the indirect utility function as the appropriate form of the utility function when observations are taken on prices. Manning and Morgan [29] have specifically applied the indirect utility function to a search problem.

p_1^L - a lower bound on selling prices for commodity 1.

p_1^U - an upper bound on selling prices for commodity 1.

The range of the dispersion of prices for commodity 1 is bounded by $(p_1^U - p_1^L)$. If $p_1^U = p_1^L$ there is no price dispersion and no search problem. Hence $p_1^U > p_1^L$. In addition the usual neo-classical assumption of non-negative prices will be made and retained throughout the thesis. In particular, it is assumed that none of the dispersed prices for commodity 1 are negative.

Assumption: $p_1^L \geq 0$.

1-4-8

$\{T_1, T_2, \dots, T_j, \dots\}$ - a sequence of time periods. A searcher may make any number of observations on p_1 within any of these time periods. At the start of a period the searcher must decide the number of observations (perhaps zero) to make within the period. In particular, at the start of period T_1 the searcher must make his market entry decision. If the choice was to enter the market, then at the start of period T_2 the searcher will utilise the prices observed (if any) in period T_1 to decide the number of observations to be taken in period T_2 . This decision sequence will continue until the searcher chooses to exit from the market. In any adaptive decision process the manner in which the flow of information from observations affects the searcher influences his optimal strategy. Obviously the searcher can be affected in many ways. Throughout the thesis the following choice is made.

Assumption: The information from and costs of observations on p_1

taken in any period reach the consumer only at the end of that period.

1-4-9

α - the discount rate per period.

p_1^{ik} - the consumer's k th observation on p_1 (offered selling price) in period T_i . For simplicity it is always assumed that observations on p_1 are statistically independent.

Assumption: All observations on p_1 are independently distributed. 1-4-10

Each seller quotes only a single price to each searcher who contacts him. However, different sellers' quotes are usually different so, from the searcher's point of view, it appears as if each seller quotes randomly from his own p.d.f. of selling prices. (1-4-10) is an assumption that the searcher believes these p.d.f.'s to be statistically independent.

c_{ik} - the (marginal) financial cost to the consumer of taking observation p_1^{ik} .

K_{ik} - the (marginal) psychic cost to the consumer of taking observation p_1^{ik} . In Chapters 5 and 6 it is assumed that the net utility expected from search by a consumer can be expressed as the difference between the utility he expects to enjoy at the end of his search and the psychic costs he incurs during his search.

Assumption: The consumer's expected indirect utility function 1-4-11

is additively separable w.r.t. expected psychic search

costs i.e. $E[I'(p_1, \dots, p_g, M-c, K)] = E[I(p_1, \dots, p_g, M-c)] - E[K]$

where I' is the consumer's indirect utility function inclusive of psychic search costs and I is the consumer's indirect utility function exclusive of psychic search costs.

t_{ik} - the transaction cost incurred by the consumer when purchasing the quantity of commodity 1 demanded from the k th seller contacted in the i th period.

Assumption: t_{ik} is independent of the quantity of commodity 1 purchased. 1-4-12

In this thesis a transaction cost is regarded as a fixed cost incurred only if a purchase of commodity 1 is made. Alternative schemes are possible. A fixed transaction cost could be incurred even if the searcher chooses to buy zero units of commodity 1. For instance, a searcher may have travelled across town during his search and, at the end of his search, could be faced with the cost involved in returning home. A second alternative is for the transaction cost to be dependent on the quantity of commodity 1 demanded. Sales taxes and stamp duties are common examples of transaction costs of this form.

$v = (v_1, v_2, \dots, v_j, \dots)$ - an observation number rule. v is a function of observed prices,

$$v : [p_1^L, p_1^U] \times [p_1^L, p_1^U] \times \dots \rightarrow I^+ \times I^+ \times \dots \quad 1-4-13$$

where I^+ is the set of non-negative integers. At the end of period

T_i , $i = 0, 1, \dots, j, \dots$ the consumer will use v to decide upon a vector

$(n_{i+1}, n_{i+2}, \dots)$ where $n_k \geq 0$ is the (integer) number of observations

on p_1 he intends, at the end of period T_i , to take in period T_k ,

$k = i+1, i+2, \dots$ The values of n_{i+1}, n_{i+2}, \dots will depend upon the values of the searcher's previous observations $p_1^{11}, \dots, p_1^{1n_1}, \dots, p_1^{i1}, \dots, p_1^{in_i}$.

At the end of period T_{i+1} the searcher will use these observations and

any additional observations $p_1^{i+1,1}, \dots, p_1^{i+1,n_{i+1}}$ taken in period T_{i+1} to decide upon a vector $(n_{i+2}, n_{i+3}, \dots)$ and so on.

y_j - the vector of observations taken on p_1 over periods T_1, \dots, T_j

$$y_j = (p_1^{11}, \dots, p_1^{1n_1}, \dots, p_1^{j1}, \dots, p_1^{jn_j}) \quad 1-4-14$$

p_{1j}^{\min} - the smallest of the observations made on p_1 in the first j periods of the consumer's search,

$$p_{1j}^{\min} = \min\{p_1^{11}, \dots, p_1^{1n_1}, \dots, p_1^{j1}, \dots, p_1^{jn_j}\} \quad 1-4-15$$

$c(j)$ - the total financial cost incurred by the consumer in making his observations on p_1 in the first j periods of his search,

$$c(j) = \sum_{i=1}^j \sum_{k=1}^{n_i} c_{ik} \quad 1-4-16$$

Assumption: $c(0) = 0$. 1-4-17

It is always assumed that there is no financial penalty associated with not searching.

\bar{M}_j - the searcher's wealth net of financial search costs after searching through periods T_1, \dots, T_j ,

$$\bar{M}_j = M - c(j) \quad 1-4-18$$

$A_j = \{a_0, a_{11}, \dots, a_{1n_1}, \dots, a_{j1}, \dots, a_{jn_j}\}$ - the terminal action set for a

search over j periods. Terminal action a_{ik} is to purchase x_1^* units of commodity 1 from the k th seller contacted in the i th period.

Terminal action a_0 is to purchase zero units of commodity 1. The consumer may choose not to enter the market and search. In this case the terminal action set is $A_0 = \{a_0\}$ and the only terminal action possible for the consumer is to allocate all of his initial wealth M to purchases of only commodities 2 to ℓ . Commodity 1 cannot be purchased without search.

δ - a terminal decision procedure. If the searcher decides to stop searching at the end of period T_j , he uses his terminal decision procedure δ to choose a terminal action from his terminal action set A_j . The terminal action chosen will depend upon the values of the observed selling prices $p_1^{11}, \dots, p_1^{1n_1}, \dots, p_1^{j1}, \dots, p_1^{jn_j}$. δ is therefore a function of observed prices,

$$\delta : [p_1^L, p_1^U] \times \dots \times [p_1^L, p_1^U] \rightarrow A_j \quad 1-4-19$$

δ is the procedure used by the searcher to determine the seller (if any) from whom commodity 1 is demanded and to determine the quantity demanded.

$\xi = (\xi_0, \xi_1, \dots, \xi_j, \dots)$ - a stopping rule. The searcher uses his stopping rule ξ at the end of a period to choose between stopping his search at this point or carrying his search on through the next period. This choice is dependent on observed prices.

At the end of the i th period T_i , $i = 0, 1, \dots, j, \dots$ ξ_i is used to determine a value S_i where $S_i = 0$ or 1 . Hence,

$$\xi : [p_1^L, p_1^U] \times [p_1^L, p_1^U] \times \dots \rightarrow \{0, 1\} \times \{0, 1\} \times \dots \quad 1-4-20$$

S_i takes values of 0 or 1 with the following meanings.

- (i) If search has persisted over periods T_1, \dots, T_i and $S_i = 0$ then the searcher's decision at the end of period T_i is to carry his search on throughout period T_{i+1} .
- (ii) If search has persisted over periods T_1, \dots, T_i and $S_i = 1$ then the searcher's decision at the end of period T_i is to stop his search now, at the end of period T_i .
- (iii) If search ceases before period T_i is reached (at least one $S_k = 1$, $k < i$) the value of S_i is of no importance.

It must be emphasised that, in general, a searcher does not know with certainty over how many periods his search will persist until the instant he makes his decision to stop searching.

- (iv) If $S_i = 1$ at the end of period T_i (for any $i = 0, 1, \dots, j, \dots$) then search cannot continue past the end of period T_i . Consequently no observations on p_1 can be taken in periods T_{i+1}, T_{i+2}, \dots . Hence $S_i = 1$ implies $n_k = 0$ for all $k \geq i+1$. Conversely, if $S_i = 0$ then search is continued throughout period T_{i+1} . This is rational in the presence of non-negative marginal search costs only if at least one observation is made in a period subsequent to period T_i . Hence $S_i = 0$ implies that at least one $n_k > 0$ where $k \geq i+1$.

$\Psi = (\psi_0, \psi_1, \dots, \psi_j, \dots)$ - a vector with components ψ_j where ψ_j is a

function of observed prices y_j with the property that $\psi_j(y_j) = \psi_j = 0$ or 1 only. $\psi_j = 1$ if and only if search is halted at the end of period T_j . $\psi_j = 0$ otherwise. Therefore, $\psi_j = 1$ is equivalent to the situation of $S_0 = S_1 = \dots = S_{j-1} = 0$, $S_j = 1$. ψ_j may therefore be expressed as the following function of S_0, \dots, S_j

$$\psi_j = (1-S_0)(1-S_1) \dots (1-S_{j-1})S_j \text{ for all } j = 0, 1, \dots \quad 1-4-21$$

$\rho = (\xi, \nu, \delta)$ - a search rule. The triple notation is a convenient way

of expressing the fact that a search rule has three components - a stopping rule ξ , an observation number rule ν and a terminal decision procedure δ . The objective of any searcher is assumed to be the maximisation of his ex ante expectation of the maximum utility attainable after search. The search rule which defines the search strategy achieving this aim is called 'optimal'.

Definition: The optimal search rule $\rho^* = (\xi^*, \nu^*, \delta^*)$ is the search rule maximising the searcher's ex ante expectation of the maximum utility attainable from search. 1-4-22

The optimal search rule is generally an adaptive rule although its behaviour in certain problems coincides with the behaviour of other search rules eg. the Stigleresque and sequential search rules. For a detailed and well referenced discussion on sequential search rules see De Groot [9, p.272] and Yahav [56]. The adaptive search rule is an application of Bellman's principle of optimality which can be loosely stated as "the optimal policy from the present state is independent of how the present state was arrived at". At each decision point the searcher must decide if he should continue searching for at least one more period. If he decides to continue searching, then as part of that decision he must have decided upon the number of observations on p_1 to take in the next period. If he decides to stop searching he must choose

a terminal action. The adaptive search rule has the form that the searcher should continue his search if he expects to gain by doing so, regardless of how fruitful or fruitless his search up to this point has been. Stigleresque search rules are thus optimal only in search problems for which ρ^* gives

$$S_0^* = 0, S_1^* = 1, n_1^* \geq 1 \quad 1-4-23$$

and sequential search rules are optimal only in search problems for which ρ^* gives

$$S_0^* = 0, \text{ and } n_i^* = 1 \text{ for all } i \geq 1 \text{ such that } S_1^* = \dots = S_i^* = 0 \quad 1-4-24$$

$V_0(\xi, \nu, \delta)$ - the present value of the searcher's ex ante expected net

utility from using a search rule $\rho = (\xi, \nu, \delta)$. (1-4-22) defines the optimal search rule $\rho^* = (\xi^*, \nu^*, \delta^*)$ as the search rule which maximises the present valued ex ante expected net utility from search.

Hence ρ^* must have the property that

$$V_0(\xi^*, \nu^*, \delta^*) \geq V_0(\xi, \nu, \delta) \quad 1-4-25$$

for any search rule $\rho = (\xi, \nu, \delta)$.

$V_j(\xi, \nu, \delta)$ - the present value, at the end of period T_j , of the

expected net utility from search conducted according to the search rule $\rho = (\xi, \nu, \delta)$. It is shown in Chapters 2 and 6 that

$V_j(\xi, \nu, \delta)$ is the greater of the utility attainable at the end of period T_j and the searcher's present valued expectation of the net utility attainable from continuing search past the end of period T_j according to the search rule $\rho = (\xi, \nu, \delta)$.

$f(p_1|w)$ - the probability density function (p.d.f.) of selling prices

for commodity 1. f describes the relative frequencies with which different values of $p_1 \in [p_1^L, p_1^U]$ are quoted by sellers. These frequencies are dependent upon w , a vector of parameters conditioning f .

Assumption: All sellers of commodity 1 are passive.

1-4-26

(1-4-26) is invoked throughout the thesis and means that any particular seller offers the same selling price to all searchers who contact him. This makes it impossible for a searcher to derive any gain by asking a seller for more than one quotation. The searcher therefore views himself as taking observations without replacement from a large population of prices p_1 where $f(p_1|w)$ describes the probabilities of observing the various selling prices for commodity 1 offered in the market place. In all sections of the thesis, except Section 3-5, it is assumed that the searcher is ignorant of any differences in sellers' relative pricing behaviour. Consequently the searcher has no choice but to regard all sellers as being identical in their pricing behaviour. In Section 3-5 the searcher is assumed to possess some imperfect information about which sellers are more likely to offer lower prices than others. This information is utilised in determining the order in which the searcher contacts sellers.

$F(p_1|w)$ - the cumulative density function (c.d.f.) of selling prices

for commodity 1,

$$F(p_1|w) = \int_{p_1^L}^{p_1} f(x|w) dx \quad 1-4-27$$

$g(w)$ - the p.d.f. describing the consumer's ex ante beliefs about the relative likelihoods of different values of w .

The consumer has complete knowledge of the functional form of $f(p_1|w)$ but, throughout all of this thesis, except Section 4-5, it is assumed that the searcher is uncertain of the actual value of w and, therefore, uncertain of the actual p.d.f. of selling prices $f(p_1|w)$. However, as observations are taken on p_1 the searcher can refine his prior p.d.f. $g(w)$ by using Bayesian pre-posterior analysis and so reduce his uncertainty

of w and $f(p_1|w)$. A short description of Bayesian pre-posterior analysis is given in Appendix 2. In Section 4-5 it is assumed that the searcher knows the true value of w and therefore knows the actual p.d.f. of selling prices $f(p_1|w)$. This assumption is equivalent to assuming $g(w)$ is degenerate at the correct value of w .

$g(w|y_j)$ - the p.d.f. describing the consumer's beliefs about the

relative likelihoods of different values of w after searching through periods T_1, \dots, T_j and making observations $p_1^{11}, \dots, p_1^{jn_j}$. This is the consumer's posterior p.d.f. on w and is related to his ex ante prior p.d.f. $g(w)$ in a manner described in Appendix 2.

$f_g(p_1^{j+1,1}, \dots, p_1^{j+1,n_{j+1}}|y_j)$ - the p.d.f. describing the consumer's

beliefs about the relative likelihoods of the values of the observations to be taken in the next period, given observations $p_1^{11}, \dots, p_1^{jn_j}$ have already been taken. This p.d.f. is described more fully in Appendix 2.

$f_g(p_1^{11}, \dots, p_1^{jn_j})$ - the p.d.f. describing the consumer's ex ante

beliefs about the values of the observations to be taken in periods T_1, \dots, T_j . This p.d.f. is described more fully in Appendix 2.

i_1, \dots, i_j - a search path. Search paths are discussed at length in

Chapter 3 which expands on the observation that rational search is not a random walk type of process. Instead it is directed by desires of maximising the gains from search by, for example, contacting sellers more likely to provide acceptable offers before contacting sellers less likely to do so. It is assumed that all sellers have unique identifying indexes. If $i_k = n$ then the meaning is that a searcher who follows the search path $i_1, \dots, i_k, \dots, i_j$ makes his k th contact with the seller whose index is n . Search paths can be revised as search proceeds. This is pursued further in Chapter 3. For the

present it is sufficient to recognise the sequence i_1, \dots, i_j as an ordered sequence of seller indexes.

c_{i_k} - the (marginal) financial cost of making an observation on the k th seller on the search path i_1, \dots, i_j , $k = 1, \dots, j$.

t_{i_k} - the transaction cost incurred by the consumer when purchasing the quantity of commodity 1 demanded from the k th seller on the search path i_1, \dots, i_j , $k = 1, \dots, j$.

w_{i_k} - the value of w ascribed by the searcher to the k th seller on the search path i_1, \dots, i_j , $k = 1, \dots, j$. The searcher's beliefs about the relative likelihoods of the value of the selling price offered by the seller with index i_k are described by the p.d.f. $f(p_1|w_{i_k})$. Recall from the above discussion of $f(p_1|w)$ that in all but Section 3-5 it is assumed that a searcher cannot distinguish differences between sellers' pricing behaviour. This is equivalent to assuming that the searcher ascribes the same value of w to all sellers.

Assumption: $w_{i_k} = w$ for all i_k . 1-4-28

In Section 3-5 assumption (1-4-28) is replaced by the assumption that the searcher has information which allows him to ascribe different values of w to different sellers and, in this way, to differentiate those sellers more likely to offer relatively high prices from those more likely to offer relatively low prices.

The above notation is now used to formally express some of the restrictions used in previous search models (listed in Section 1-3) as options which can be built into a search model. Alternatives to these restrictions are listed alongside as alternative options.

Option 1(a): The demand for commodity 1, x_1^* , is fixed before search experience is available and before the length of search is known.

The value of x_1^* is not revised as search proceeds.

Option 1(b): x_1^* may be revised as search proceeds.

Option 2(a): There are financial costs of search ie. $c > 0$.

Option 2(b): There are no financial costs of search ie. $c \equiv 0$.

Option 3(a): There are psychic costs of search ie. $K > 0$.

Option 3(b): There are no psychic costs of search ie. $K \equiv 0$.

Option 4(a): The consumer's indirect utility function is linear in the price of commodity 1 so that

$$E[I(p_1, p_2, \dots, p_\ell, \bar{M})] = I(E[p_1], p_2, \dots, p_\ell, \bar{M}) \quad 1-4-29$$

Option 4(b): The consumer's indirect utility function need not be linear in p_1 .

Option 5(a): Search lasts for one period and at least one observation on p_1 is taken; $S_0^* = 0$, $S_1^* = 1$, $n_1^* \geq 1$ (Stigleresque search).

Option 5(b): Search lasts for at least one period and exactly one observation on p_1 is taken in each of these periods; $S_0^* = 0$, and $n_i^* = 1$ for every $i \geq 1$ such that $S_1^* = \dots = S_i^* = 0$ (sequential search).

Option 5(c): Search may last for any number (including zero) of periods and any number (including zero) of observations on p_1 may be taken in any period (generalised search). It should be noted that this option includes both the Stigleresque and sequential search options 5(a) and 5(b) as special cases.

The options can now be used to categorise the search models presented in earlier work and in this thesis.

Stigler's original consumer search model [50] relies on options of fixed demand, 1(a), utility linear in p_1 , 4(a), search behaviour option 5(a) and, although his discussion makes it clear that psychic search costs are important, his analysis includes only financial search costs - options 2(a) and 3(b).

The sequential search models of Rothschild [43], Gastwirth [11], Telser [52], and Axell [1] can be categorised as containing the options of fixed demand, 1(a), utility linear in p_1 , 4(a), search behaviour option 5(b) and containing only financial costs of search, 2(a) and 3(b).

The indirect utility approach to search allows immediate and natural generalisations of both of the above types of models. In Chapters 5 and 2 respectively the basic Stigleresque and sequential search models outlined above are generalised by replacing options 1(a), 3(b) and 4(a) by options 1(b), 3(a) and 4(b) ie. the fixed demand restriction is replaced by the option allowing variable demand, psychic search costs are specifically included with financial search costs and the consumer's utility function is no longer restricted to being linear in p_1 . Manning and Morgan [29] forms the basis of the analysis of Chapter 5.

Chapter 6 contains an analysis of the most general possible combination of options 1 to 5 ie. 1(b), 2(a), 3(a), 4(b) and 5(c). Demands may vary continuously, both psychic and financial search costs may be incurred, the searcher may have any cardinal utility function that is additively separable w.r.t. psychic costs and the searcher is permitted a generalised search rule which includes both Stigleresque and sequential search rules.

It is of course possible to add more options to the above list. For example, a series of assumptions used in the thesis was listed earlier in this section. There is nothing to prevent these assumptions being listed with their alternatives as options also. The theorist must choose. There are two principal reasons for the choice of options made in this thesis. First, the options fixed as assumptions are the same as those of previous authors. One of the contributions of this

thesis is the use of the models developed in this thesis for a critical examination of these authors' models. A comparison such as this is valid only if the models in this thesis contain the same axiomatic base as the earlier models. Second, previous authors chose this base principally because it offers relative statistical simplicity and makes the economics of information contained in consumer search models easier to extract. Nevertheless, the additional restrictions imposed by earlier authors severely limits the amount of economic analysis possible in their models. The generalisations made in this thesis substantially increase this yield.

Each possible combination of the above options represents a different variation on the search problem theme. It is convenient to collect the variations into four groups.

Type 1 Variations: No Search Costs:

Price information is both financially and psychically free in search models containing both option 2(b) and option 3(b). The consumer will continue to search until he finds a seller offering the lowest price because, without search costs, his expected net gain from search is always positive until the lowest price is attained. Any initial price dispersion must collapse to the lowest price in these variations which are therefore both unrealistic and uninteresting.

Proposition 1-4-1:

If $c \equiv K \equiv 0$, then no consumer's search terminates until the consumer's lowest observed price p_1^{\min} is the lowest market price.

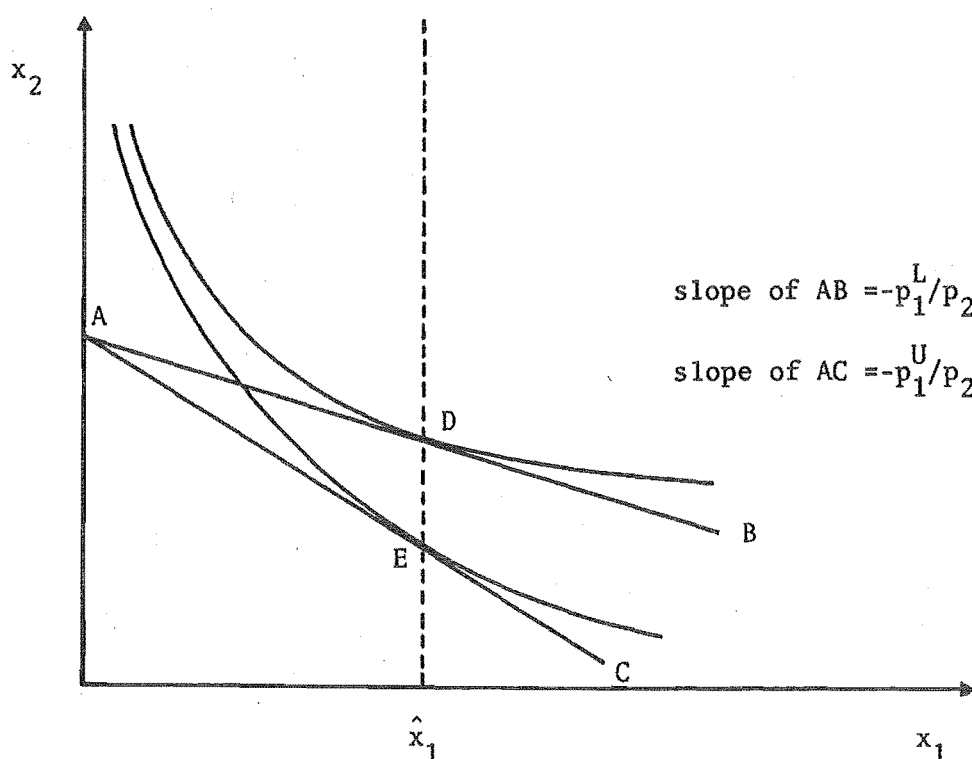
Type 2 Variations: Psychic Search Costs, No Financial Search Costs, and Fixed Demand for Commodity 1:

These variations contain options 1(a), 2(b) and 3(a). The absence of financial search costs assures the searcher that his wealth at any

stage of his search is unchanged from his initial wealth M . For the moment assume positive marginal psychic costs are sufficient to prevent search continuing for ever.[†]

The restriction placed upon the class of utility functions admissible to search problems with a fixed demand constraint $x_1^* = \hat{x}_1$, where $\hat{x}_1 > 0$ is a constant, is illustrated in Figure 1-4-1. The range

Figure 1-4-1



of possible values of p_1^{\min} , $[p_1^L, p_1^U]$, defines a set of possible budget constraints with bounds AB and AC. Since \hat{x}_1 is the quantity of commodity 1 demanded by the consumer whatever value of p_1^{\min} is the outcome from search, the consumer's price expansion path as p_1^{\min} varies must be ED.

The class of utility functions for which $x_1^* = \hat{x}_1$ for any value of $p_1^{\min} \in [p_1^L, p_1^U]$ is clearly very special. Solving for these functions

[†] This sufficiency is proved in Chapters 5 and 6

requires the solution of a simultaneous system of $(\ell-1)$ partial differential equations derived from the 1st-order utility maximisation conditions. This task is difficult. Recall that the consumer's demand function for commodity 1 is parameterised by p_2, \dots, p_ℓ, M ie.

$x_1^* = x_1(p_1^{\min}, p_2, \dots, p_\ell, M)$. The fixed demand hypothesis is that

$$x_1(p_1^{\min}, p_2, \dots, p_\ell, M) = \hat{x}_1 \text{ for all } p_1^{\min} \in [p_1^L, p_1^U] \quad 1-4-30$$

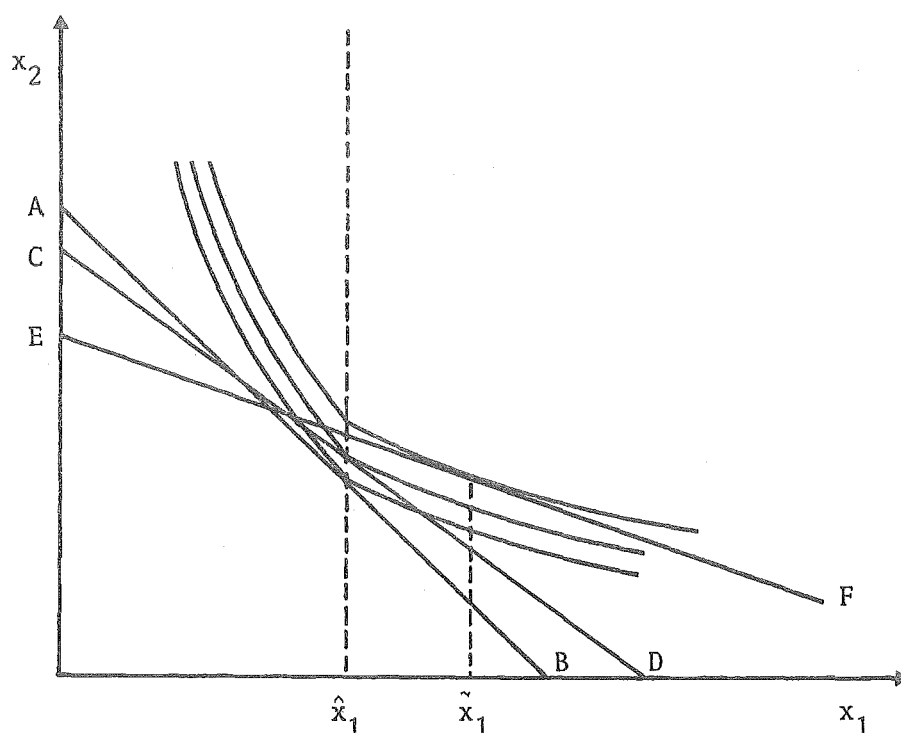
The utility functions for which (1-4-30) is true are therefore parameterised by particular values of p_2, \dots, p_ℓ, M for a given value of \hat{x}_1 . It follows that, if the consumer's utility function is smooth, altering the value of any one of p_2, \dots, p_ℓ, M will alter the quantity of commodity 1 demanded from \hat{x}_1 . This means that comparative statics analysis is impossible under the fixed demand hypothesis if the consumer's utility function is smooth.

Proposition 1-4-2:

Under the fixed demand hypothesis there is no smooth utility function for which comparative statics analysis is possible.

The proposition rules out comparative statics analysis for these search problems only if the utility functions are smooth. Comparative statics analysis may still be possible if the utility functions are not smooth, but only for changes in p_2, \dots, p_ℓ, M within limits which depend upon the degree of non-smoothness of the utility functions. As an example, see Figure 1-4-2 which demonstrates the two commodity case ($\ell=2$) where the consumer's preferences are not smooth at $x_1 = \hat{x}_1$.

Figure 1-4-2



A decrease in the price of commodity 2 from p_2^1 to p_2^2 alters the budget constraint from AB to CD but does not violate the condition $x_1^* = \hat{x}_1$. A decrease from p_2^1 to p_2^3 alters the budget constraint from AB to EF, alters the consumer's demand for commodity 1 from \hat{x}_1 to \tilde{x}_1 and, in doing so, violates the condition $x_1^* = \hat{x}_1$. Comparative statics analysis may therefore be possible for parameter changes within some interval. The width of this interval is determined by the degree of non-smoothness possessed by the utility function at $x_1 = \hat{x}_1$. Let

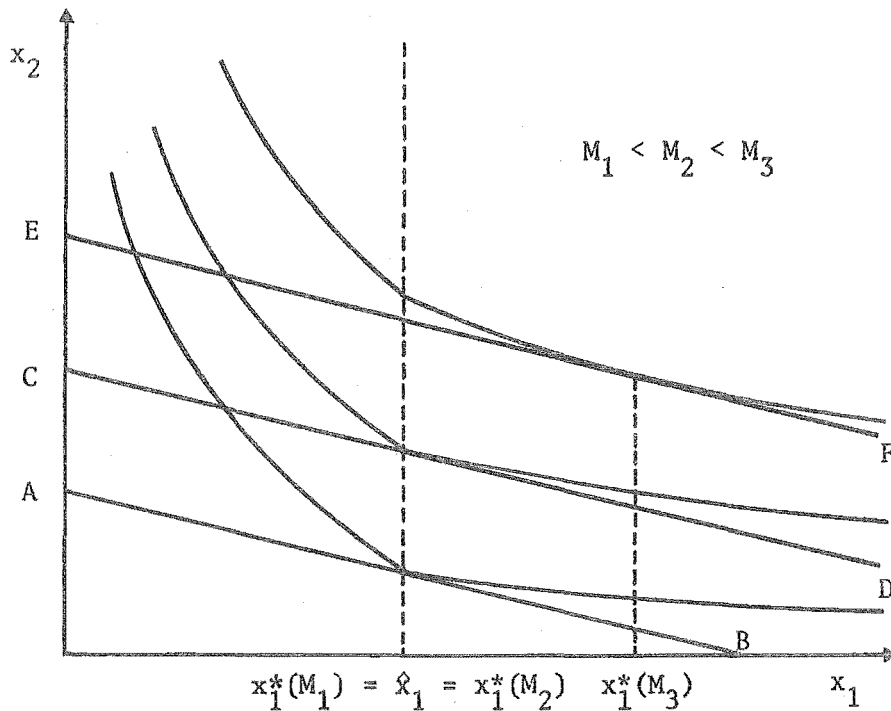
$$\lim_{x_1 \rightarrow \hat{x}_1^-} (\partial U / \partial x_1) / (\partial U / \partial x_2) = \alpha, \quad \lim_{x_1 \rightarrow \hat{x}_1^+} (\partial U / \partial x_1) / (\partial U / \partial x_2) = \beta \quad 1-4-31$$

where $\alpha > \beta$. The degree of non-smoothness of the utility function at $x_1 = \hat{x}_1$ can be measured by the difference $(\alpha - \beta)$. Comparative statics analysis can be carried out for changes in p_2 only from p_2' to p_2'' where both p_2' and $p_2'' \in (p_1^{\min} / \alpha, p_1^{\min} / \beta)$. Similar constraints exist on com-

parative statics results for changes in M .

An example of this is shown in Figure 1-4-3 where increasing M from M_1 to M_2 to M_3 results in the budget constraint rising from AB to CD or to EF and the consumer's demand for commodity 1 increasing from \hat{x}_1 to $x_1^*(M_3)$.

Figure 1-4-3



If \hat{M} is the value of M such that

$$\lim_{M \rightarrow \hat{M}} \lim_{x_1 \rightarrow x_1^*} (\partial U / \partial x_1) / (\partial U / \partial x_2) = p_1^{\min} / p_2 \quad 1-4-32$$

then comparative statics analysis can be carried out for changes in M only within the interval $[0, \hat{M}]$.

Type 3 Variations: Financial Search Costs and Fixed Demand for

Commodity 1:

These variations simultaneously contain options 1(a) and 2(a).

In Section 1-3 it was noted that restrictions of these two types are

common in the search literature eg. [1], [3], [11], [43], [50], [52], [53]. Now note that the effect of financial search costs is to alter the searcher's wealth from M to $\bar{M} < M$. It is an immediate implication of Proposition 1-4-2 that the consumer's utility function cannot be smooth in these search problems.

Proposition 1-4-3:

If search is financially costly and $x_1^* = \hat{x}_1$ for all $p_1^{\min} \in [p_1^L, p_1^U]$, then the consumer's utility function is not smooth at $x_1 = \hat{x}_1$.

Proof:

The consumer's demand for commodity 1 is $x_1^* = \hat{x}_1$ for any price-income situation. Let j_1 and j_2 be two different search lengths. The wealth of the consumer after search is $\bar{M}_{j_i} = M - c(j_i)$ for search length j_i , $i = 1, 2$. Denote the minimum observed price p_1^{\min} for a search length of j_i by $p_{1j_i}^{\min}$ for $i = 1, 2$. Let $\hat{x}_1, x_2^*, \dots, x_\ell^*$ be the optimal allocation of net wealth \bar{M}_{j_1} for a consumer when search of length $j_1 < j_2$ reveals a minimum price of $p_{1j_1}^{\min}$. A sufficient but not necessary condition for search of length j_2 to be undertaken is that there is a price $p_{1j_2}^{\min}$ such that the allocation $\hat{x}_1, x_2^*, \dots, x_\ell^*$ can be attained with the smaller net wealth \bar{M}_{j_2} . Suppose such a price $p_{1j_2}^{\min}$ exists. The $(\ell-1)$ dimensional hyperplanes defined by the two budget constraints are

$$p_{1j_i}^{\min} x_1 + p_2 x_2 + \dots + p_\ell x_\ell = \bar{M}_{j_i} \quad \text{for } i = 1, 2 \quad 1-4-33$$

These two hyperplanes must intersect along the $(\ell-2)$ dimensional hyperplane defined by setting $x_1 = \hat{x}_1$ in (1-4-33),

$$p_2 x_2 + \dots + p_\ell x_\ell = \bar{M}_{j_i} - p_{1j_i}^{\min} \hat{x}_1 \quad \text{for } i = 1, 2 \quad 1-4-34$$

For search of length j_1 or j_2 the optimal consumptions must lie in the hyperplane (1-4-34). For search of length j_1 the preferred consumption is $\hat{x}_1, x_2^*, \dots, x_\ell^*$. For search of length j_2 the consumer must again choose from the consumptions in the hyperplane (1-4-34). The axiom of revealed

preference (see Richter [40]) requires that the consumer again chooses the consumption $\hat{x}_1, x_2^*, \dots, x_\ell^*$. This implies two distinct hyperplanes (1-4-33) support the strictly convex set defined by the indifference hypersurface at $\hat{x}_1, x_2^*, \dots, x_\ell^*$. This is possible if and only if the indifference hypersurface is not smooth at $\hat{x}_1, x_2^*, \dots, x_\ell^*$. Q.E.D.

Figure 1-4-4

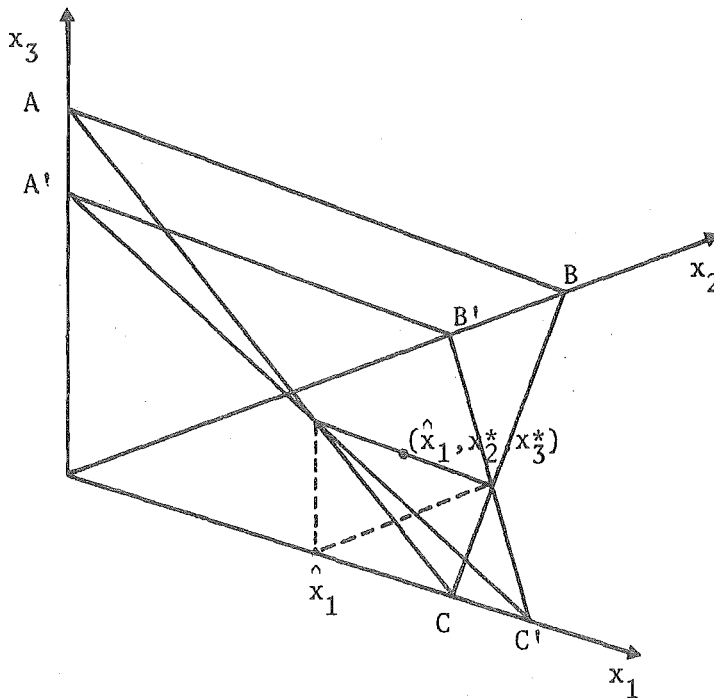


Figure 1-4-4 illustrates this result for $\ell = 3$ (the result is geometrically obvious for $\ell = 2$). ABC is the budget constraint after search of length j_1 and $A'B'C'$ is the budget constraint after search of length j_2 . Both intersect on the two-dimensional hyperplane with $x_1 = \hat{x}_1$. Since the consumption $(\hat{x}_1, x_2^*, x_3^*)$ was chosen from all others in this hyperplane for search of length j_1 , $(\hat{x}_1, x_2^*, x_3^*)$ must again be chosen from all others in this hyperplane for search of length j_2 . Consequently, the indifference hypersurface cannot be smooth at $x_1 = \hat{x}_1$ in the direction of any $x_1 x_j$ -plane, $j = 2, 3$.

Included in the discussion of Type 2 variations was the point that comparative statics analysis was possible only if the consumer's utility function was not smooth for $x_1 = \hat{x}_1$. In the Type 3 variations now being considered, the utility functions cannot be smooth either, so similar arguments apply to the use of comparative statics analysis. Comparative statics results can only be obtained for changes in any one of $p_2, \dots, p_\ell, \bar{M}$ that are between limits set by the degree of non-smoothness of the indifference hypersurface. This is particularly important when considering changes in the consumer's net wealth \bar{M} since the fixed demand condition may be violated if financial search costs accumulate to the point where \bar{M} passes outside the limits set by the indifference hypersurface. Such considerations have not been expressed by any previous researchers, the only implicit consideration being that no searcher will allow his net wealth to fall below the amount required to purchase \hat{x}_1 units of commodity 1.

The Type 3 variations containing the sequential search rule option 5(b) are analysed in detail in Chapter 2. They are also used in Chapter 3 for describing the choice of an optimal search path and in Chapter 4 for deriving transaction prices and search length probability distributions. The Type 3 variations containing the Stigleresque search rule option 5(a) or the generalised search rule option 5(c) form special cases of the models developed in Chapters 5 and 6.

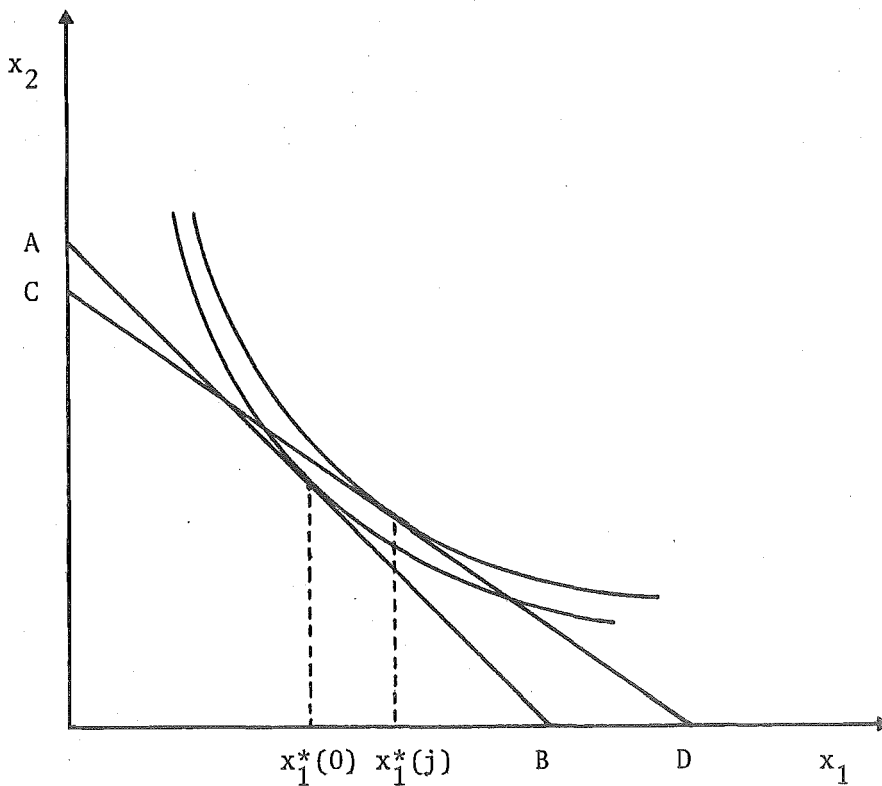
Type 4 Variations: Search Costs and Variable Demand for Commodity 1:

These variations contain options 1(b) and either one or both of options 2(a) or 3(a). Search is thus restricted by financial and/or psychic search costs and the demand for commodity 1 may vary as the least observed price p_1^{\min} and/or the consumer's net wealth \bar{M} vary.

Smooth utility functions are admissible for these search problems despite

the presence of financial search costs. Figure 1-4-5 demonstrates this for $\ell = 2$ and Stigleresque search behaviour. $x_1^*(0)$ is the demand for commodity 1 decided upon before search on the basis of the expected minimum price for commodity 1 for j observations $E[p_1^{\min}|j]$ and a post-search net wealth of \bar{M}_j . The actual least observed price p_{1j}^{\min} causes the consumer to revise his demand for commodity 1 from $x_1^*(0)$ to $x_1^*(j)$ if his utility function is smooth.

Figure 1-4-5



The variations containing the generalised search option 5(c) are left until Chapter 6. The variations containing the Stigleresque search option are analysed in Chapter 5. The (Type 4) variations containing the sequential search option 5(b) are not specifically considered for two reasons which would make their inclusion largely

repetitious. First, it would mean a repetition of the bulk of the analysis carried out in Chapter 5. Second, the generalised search rule contains the sequential search rule (and the Stigleresque search rule) as an extreme case. Type 4 variations containing the sequential search option are therefore specified in Chapter 6 as a special case.

Spot and Futures Markets;

The real world is a mixture of imperfect spot and futures markets. Yet the consumer search problem is always thought of only in the context of imperfect spot markets. Futures markets are not admitted to consumer search problems because doing so greatly complicates the analysis of these problems. If the existence of futures markets is recognised, then it must also be recognised that a searcher's actions in present spot markets are partly determined by his beliefs on the likelihoods of various outcomes possible for future events. Consequently, analysis of consumer search problems would have to be extended to provide descriptions of the manner in which a consumer allocates wealth to futures commodities for both present and future futures markets, descriptions of the consumer's expenditure patterns in the imperfect spot markets of the future, descriptions of the consumer's assessment of the likelihoods of the outcomes of various future events and how these different outcomes will influence markets at times when the consumer's various futures contracts fall due.

The purpose of this section has been to formally distinguish between some of the search problems considered in the literature. Deficiencies in some of their analyses have been pointed out together with some of the more subtle consequences of restrictions commonly imposed on consumer search models. Chapters 2 to 6 extend the models

labelled as Types 3 and 4. Type 3 models are examined in Chapters 2, 3 and 4. Type 4 models are examined in Chapter 5. Chapter 6 presents the generalised search model and demonstrates how it encompasses both Stigleresque and sequential search models.

CHAPTER II

SEQUENTIAL STOPPING RULES APPLIED TO SEARCH PROBLEMS WITH FIXED ORDER QUANTITY RESTRICTIONS

SECTION 2-1: ASSUMPTIONS ON SEARCHERS' KNOWLEDGE AND SEARCH VARIABLES IN THE ROTHSCCHILD, TELSER, GASTWIRTH AND AXELL SEARCH MODELS

In Chapter 1 the search models analysed by Rothschild [43], Telser [52], Gastwirth [11] and Axell [1] were categorised as all containing only financial search costs and a fixed demand for commodity 1. Additional restrictions contained in these models are:

- (i) minimisation of expenditure on commodity 1 is equivalent to maximisation of consumer utility,
- (ii) there are no transaction costs,
- (iii) the conditions of search are such that sequential search rules are optimal ie. it is optimal to make exactly one observation on p_1 in each period over which search persists,
- (iv) the marginal financial cost of search is the same for each observation on p_1 ,
- (v) search always ends in the purchase of a unit of commodity 1,
- (vi) the market entry decision of the consumer must always be to enter the market and begin searching.

The claims made by these authors are that their models are comparable to Stigler's consumer search model [50] and that their use of 'optimal' sequential search rules remedies a substantial deficiency in Stigler's model. However, the discussion in Chapter 1 makes two points clear:

- (i) the absence of psychic costs in the above models means they are not directly comparable to Stigler's model. In particular, minimisation of expenditure on commodity 1 is not necessarily equivalent to utility

maximisation in the presence of psychic costs. The conditions under which minimisation of expenditure on commodity 1 is equivalent to utility maximisation are discussed in Section 2-6.

(ii) Sequential search rules are optimal only when the nature of the problem permits it. Neither Stigler nor the above authors considered what conditions might be necessary and/or sufficient in search problems for their two respective search rules to be the optimal search rule. In Chapter 6 these two sets of conditions are shown to be limiting on a more general set of conditions on search problems.

The sequential search rules used by Rothschild, Telser, Gastwirth and Axell all have the following simple form. A searcher must make $j \geq 1$ observations on p_1 , the least of which is p_{1j}^{\min} . The searcher chooses to make his $(j+1)$ th observation on p_1 if and only if

$$p_{1j}^{\min} - E[p_{1,j+1}^{\min} | p_{1j}^{\min}] > c \quad 2-1-1$$

where c is the constant marginal financial cost of search. The rule expresses the idea that the next observation is taken if and only if the searcher expects to lower his expenditure on commodity 1 (inclusive of financial search costs) by doing so. If (2-1-1) is not satisfied, search is stopped and a single unit of commodity 1 is purchased free of transaction costs from the seller who offered the lowest price.

The stopping rule (2-1-1) has a particular simplifying structure called a 'supermartingale' which makes the rule 'myopic' in that the searcher need consider only his expectations about the next observation (decision point) and need not consider expectations about observations beyond the next. Supermartingales are discussed in Sections 2-6 and 3-7 and in Appendix 3. The reader is referred to De Groot [9, p.353] and McCall [30, p.423] for detailed descriptions of martingales.

Assumptions on search costs and searchers' knowledge of prices

implicit in the use of a search rule as simple as (2-1-1) are set out below. The severity of these assumptions largely depends on the physical manner in which a searcher carries out his search.

Assumption: The searcher has no knowledge of sellers' relative pricing behaviour. 2-1-2

Assumption: The marginal financial search cost of any observation on p_1 is a constant c and is independent of any characteristic of any other observation on p_1 . 2-1-3

Assumption: There are no transaction costs. 2-1-4

Consider assumption (2-1-4) first. The searcher incurs a transaction cost only if he purchases a unit of commodity 1. Furthermore, a transaction cost is incurred only at the completion of search when a purchase is made. If transaction costs are present, the seller offering the lowest observed selling price need not be the seller from whom a unit of commodity 1 can be obtained with least expenditure. For example, if seller A has offered a selling price of \$5 and seller B a price of \$6, and their respective associated transaction costs are \$3 and \$1, then seller B is preferred to seller A. The presence of different transaction costs associated with different sellers therefore means that the expenditure minimising terminal action need not be to buy from the seller who offered the lowest price.

Assumption (2-1-2) simplifies the statistical tools required to analyse these search models by allowing the use of the same p.d.f. of selling prices $f(p|w)$ in determining successive expectations on minimum observed selling prices $E[p_{1,j+1}^{\min} | p_{1j}^{\min}]$ in the stopping rule (2-1-1). In real life a searcher may observe signals from sellers which cause him to believe there are differences in sellers' relative pricing behaviour, and so alter the searcher's sequence of expected minimum

prices. For example, a hotel with a uniformed doorman may be expected to charge a higher tariff for its rooms than one without a doorman. Similarly, experience has taught people that a jar of coffee usually costs more if it is bought from a small corner store instead of a large discount supermarket. Assumption (2-1-2) is relaxed in Section 3-5 where searchers' knowledge of sellers' relative pricing behaviour is discussed as a factor in determining a searcher's optimal search path.

Assumption (2-1-3) is more sensitive to the physical manner in which search is conducted than are assumptions (2-1-2) and (2-1-4). It is useful to think of c_j as the marginal cost incurred by the searcher's communicating with the j th seller on his search path. $c_j > 0$ so communication is never free. The word 'communication' is used in a wide sense to include a variety of physical search methods. A searcher may communicate with sellers by travelling from one seller to another to communicate on a face to face basis, or he may communicate by telephoning sellers, or he may communicate by simply reading sellers' advertisements. The method employed has a large bearing on the magnitude of the search costs - an enquiry over the telephone usually costs less than a journey to enquire in person. If the method of search chosen is to travel from one seller to another, the cost of the next observation on p_1 is clearly dependent on where the last observation on p_1 was taken. The distance separating the next seller from the present seller's position depends upon the present seller's position. In such cases the marginal cost of the next observation is strongly dependent on at least one characteristic (location) of another observation, and it is unlikely that the marginal financial costs of different observations will be the same. If the method of search chosen is telephone enquiries, sellers' locations will

be less important characteristics than the characteristic of whether or not particular sellers have telephones. This characteristic is unlikely to affect the marginal financial cost of any other observation on p_1 and the financial cost per telephone call is likely to be a constant. Assumption (2-1-2) is therefore less severe for telephone searches than for face to face searches.

The most serious aspect of the simplistic rule (2-1-1), however, is that its use could contradict the hypothesis of fixed demand for commodity 1, $x_1^* = 1$, upon which the rule relies. The possibility is demonstrated by the following simple numerical example. Suppose all sellers offer one of only two selling prices for commodity 1, \$10 and \$1, and that

$$f(p_1|w) = \begin{cases} 0.1, & p_1 = \$10 \\ 0.9, & p_1 = \$1 \end{cases}$$

A searcher begins with an initial wealth of \$20. The constant marginal financial search cost is \$6. Suppose the compulsory first observation p_1^1 chances to be \$10. The searcher's net wealth is now \$14. The expected reduction in selling price from a second observation is

$$\$10 - (\$10 \times 0.1 + \$1 \times 0.9) = \$8.10 > \$6 \quad 2-1-5$$

so rule (2-1-1) decrees a second observation on p_1 must be taken.

Suppose the second observation chances to be \$10. The lowest observed price is $p_{12}^{\min} = \$10$ but the searcher's net wealth is now only \$8 and the fixed demand condition $x_1^* = 1$ cannot be met.

The models used by Rothschild, Telser, Gastwirth and Axell will be compared with the search model developed in Sections 2-2, 2-3, 2-4 and 2-5 which uses a sequential search rule and contains only financial search costs. The utility functions admissible for this model are

therefore characterised by a non-smoothness similar to that described in Section 1-4 for Type 3 search models. Assumption (2-1-2) and the assumption of independence of marginal costs are retained for simplicity, but the above models are extended in the following ways:

- (i) the consumer is allowed to choose whether or not to search,
- (ii) the consumer's demand for commodity 1 may be either one unit or zero units ie. the consumer may choose not to search or to exit from the market without purchasing any of commodity 1,
- (iii) the marginal financial cost c_j of the j th observation on p_1 is not restricted to be the same for all $j = 1, 2, \dots$
- (iv) consumer utility maximisation need not be coincident with minimisation of consumer expenditure on commodity 1,
- (v) transaction costs are not excluded.

In Section 2-6 this model is shown to reduce to the Rothschild etc. search models if the consumer's utility function belongs to a particularly heavily restricted class of utility functions.

It is not of particular consequence that (ii) restricts the only non-zero quantity of commodity 1 that can be demanded to one unit. The analysis can easily be extended to allow any positive integer quantity demanded for commodity 1.

SECTION 2-2: NOTATION AND AN OUTLINE OF A SEQUENTIAL SEARCH MODEL WITH ONLY FINANCIAL SEARCH COSTS AND ZERO-ONE DEMAND FOR COMMODITY ONE

For clarity, the assumptions made in the analysis of this model are stated at the outset. Additional notation is presented to allow easier comparison of this model to the models discussed in Section 2-1.

Assumption: The conditions imposed on the searcher by the search problem he faces are such that a sequential search

rule is optimal.

Assumption: The consumer chooses the search rule which

2-2-2

maximises his ex ante expected utility.

(2-2-1) requires that only a sequential search rule may be optimal.

In this model, therefore, only one observation on p_1 will be made in each period over which search persists. This permits the following notational simplifications.

(i) For a sequential search rule, the observation number rule v has the simple form of $v = (v_0, \dots, v_j, \dots)$ where

$$v_j(y_j) = (1, n_{j+2}, n_{j+3}, \dots) \text{ for all } j \text{ s.t. } S_1 = \dots = S_j = 0 \quad 2-2-3$$

Because v is fixed in this way it is convenient to drop it from the search rule notation and write a sequential search rule as

$$\rho_s = (\xi, \delta) \quad 2-2-4$$

The sequential search rule which maximises ex ante utility is therefore

$$\rho_s^* = (\xi^*, \delta^*) \quad 2-2-5$$

δ^* , the optimal terminal decision procedure component of ρ_s^* is presented in Section 2-4. ξ^* , the optimal stopping rule component of ρ_s^* is presented in Section 2-3. The optimality of ρ_s^* is proved in Section 2-5 and the conditions under which ρ_s^* simplifies to the simple rule (2-1-1) are presented in Section 2-7.

(ii) Under a sequential search rule only one observation on p_1 is made in each period over which search persists. In the notation presented in Section 1-4, the index i refers to the period number and the index k refers to the number of the observation taken within a period. Hence, under a sequential search rule, $k = 1$ only for each period over which search persists and consequently it is convenient to drop it from the notation while only sequential search rules are being considered. The i th sequential observation on p_1 is p_1^{i1} , the marginal financial cost of

taking it is c_{i1} and the associated transaction cost is t_{i1} . These are rewritten as p_1^i , c_i and t_i . This notational form is akin to the notation used in earlier sequential search models. The notational modifications are

$$p_1^i = p_{i1}^{i1} \quad 2-2-6$$

$$c_i = c_{i1} \quad 2-2-7$$

$$t_i = t_{i1} \quad 2-2-8$$

$$y_j = (p_1^1, \dots, p_1^j) \quad 2-2-9$$

$$p_{1j}^{\min} = \min\{p_1^1, \dots, p_1^j\} \quad 2-2-10$$

$$c(j) = \sum_{i=1}^j c_i \quad 2-2-11$$

$$A_j = \{a_0, a_1, \dots, a_j\} \text{ where } a_i = a_{i1} \text{ for } i = 1, \dots, j. \quad 2-2-12$$

Assumption: There are no psychic costs of search ie. $K \equiv 0$ 2-2-13

Assumption: There are financial costs of search and $c_j > 0$ 2-2-14

and finite for all $j \geq 1$.

Assumption: Commodity 1 is indivisible. 2-2-15

Assumption: The consumer's demand for commodity 1, $x_1^* = 0$ 2-2-16

or 1 only.

If the consumer's demand for commodity 1 is zero units he incurs no transaction cost. If the consumer's demand for commodity 1 is one unit and it is purchased from the i th seller contacted, the payment made to the i th seller by the consumer is $(p_1^i + t_i)$. If j sellers are contacted, the minimum payment that can be made in purchasing a unit of commodity 1 from any of these sellers is

$$(p_1 + t)_j^{\min} = \min\{p_1^1 + t_1, \dots, p_1^j + t_j\} \quad 2-2-17$$

Assumption: The consumer fixes upon a particular search path 2-2-18

prior to beginning his search and does not revise it as

search proceeds.

In addition, all of the assumptions made in Section 1-4, except assumption (1-4-11), apply through this chapter. Assumption (1-4-11) is not necessary since assumption (2-2-13) is that psychic search costs do not exist. Some additional notational modifications are now presented.

$V_0(\xi, \delta)$ - since a sequential search rule is written as $\rho_s = (\xi, \delta)$,

the consumer's ex ante expected maximum utility from using ρ_s can now be written as $V_0(\xi, \delta)$.

$V_j(\xi, \delta)$ - the searcher's expectation, after he has taken j observations on p_1 , of the utility attainable from using the sequential search $\rho_s = (\xi, \delta)$.

The optimal sequential search rule $\rho_s^* = (\xi^*, \delta^*)$ has the property that no other sequential search rule can provide the consumer with a higher ex ante expected utility ie.

$$V_0(\xi^*, \delta^*) \geq V_0(\xi, \delta) \quad 2-2-19$$

for any sequential search rule $\rho_s = (\xi, \delta)$.

TE_j - the searcher's total expenditure on commodity 1 throughout his search should he stop searching after taking j observations on p_1 ie.

$$TE_j = (p_1^i + t_i) + c(j), \text{ if the chosen terminal action is } a_i \quad 2-2-20$$

$$TE_j = c(j), \text{ if the chosen terminal action is } a_0 \quad 2-2-21$$

$W_0(\xi, \delta)$ - the searcher's ex ante expectation of the minimum total expenditure on commodity 1 attainable from using a sequential search rule $\rho_s = (\xi, \delta)$.

$W_j(\xi, \delta)$ - the searcher's expectation, after he has taken j observations on p_1 , of the minimum total expenditure on commodity 1 attainable

from using a sequential search rule $\rho_s = (\xi, \delta)$.

The sequential search rule $\hat{\rho}_s = (\hat{\xi}, \hat{\delta})$ is defined as having the property that no other sequential search rule can provide the consumer with a lower ex ante expected total expenditure on the purchase of a unit of commodity 1 ie.

$$W_0(\hat{\xi}, \hat{\delta}) \leq W_0(\xi, \delta) \quad 2-2-22$$

for any sequential search rule $\rho_s = (\xi, \delta)$.

SECTION 2-3: THE OPTIMAL STOPPING RULE FOR THE SEQUENTIAL SEARCH MODEL

Kohn and Shavell [20] are the authors of the presently definitive paper on optimal sequential stopping rules. In their analysis they recognise the indirect utility function as the utility function appropriate to search problems in which the observations are prices. Their statement of the form of the general optimal sequential stopping rule (see Section 1-2) is a paraphrase of Bellman's optimality principle since, as already mentioned, this principle underlies any type of adaptive decision procedure. The sequential stopping rule derived here is particular to the model considered and may be considered as a special case of Kohn and Shavell's more general rule. However, the rule derived here explicitly recognises the consumer's allocation problem as an integral part of the optimal stopping rule ξ^* and therefore of the optimal search rule ρ_s^* . Kohn and Shavell's analysis is sufficiently general for this recognition but they did not make it explicit, the thrust of their paper being towards other properties of sequential stopping rules.

The optimal sequential stopping rule ξ^* for the search model under consideration is derived as a 'truncated backward induction' stopping rule under the temporary assumption that there is a finite

number of sellers. This number is a finite upper bound on the number of observations that will be taken using an optimal sequential search rule. In Section 2-5 this finite upper bound is proved to always exist when financial search costs are present, making the assumption of a finite number of sellers unnecessary.

Assumption: J is a finite upper bound on the number of observations on p_1 that will be taken by a searcher using an optimal sequential search rule. 2-3-1

Derivation of the Truncated Backward Induction Stopping Rule ξ^{bJ} :

$$\xi^{bJ}, \text{ by (1-4-20), is } \xi^{bJ} = (\xi_0^{bJ}, \xi_1^{bJ}, \dots, \xi_J^{bJ}) \text{ where}$$

$$\xi_j^{bJ}(y_j) = S_j^{bJ} = \begin{cases} 0, & \text{if the } (j+1)\text{th observation on } p_1 \text{ is to be taken} \\ 1, & \text{if only } j \text{ observations on } p_1 \text{ are to be taken} \end{cases} \quad 2-3-2$$

and where

$$\xi_J^{bJ}(y_J) = S_J^{bJ} \equiv 1 \quad 2-3-3$$

since, by assumption (2-3-1), no more than J observations can be made on p_1 .

Recall from Section 2-2 that $V_j(\xi, \delta)$ is defined as being the maximum utility the consumer expects, after taking j observations on p_1 according to a search rule $\rho_s = (\xi, \delta)$, to be attainable. To denote the presence of the upper bound J on the number of observations that a consumer will make, this utility will be written as $V_j^J(\xi^J, \delta)$.

Suppose a consumer using a search rule $\rho_s^{bJ} = (\xi^{bJ}, \delta)$ has taken J observations on p_1 . His attainable level of utility, given a terminal decision procedure δ , is $U(\delta(y_J))$. Hence,

$$V_J^J(\xi^{bJ}, \delta) = U(\delta(y_J)) \quad 2-3-4$$

Now suppose a consumer has taken $(J-1)$ observations on p_1 . If

he chooses to halt his search at this point his attainable utility is $U(\delta(y_{J-1}))$. If he chooses to take an additional observation, p_1^J , the utility expected after so doing is $E[U(\delta(y_J)) | y_{J-1}]$. This expectation is conditional upon previous observations y_{J-1} for two reasons. First, the utility expected after observation p_1^J depends on the utility already made attainable by offered prices p_1^1, \dots, p_1^{J-1} . Second, the expectation is made with respect to the p.d.f. $f_g(p_1^J | y_{J-1})$, which is the searcher's assessment of the relative likelihoods of different values of p_1^J being quoted. This assessment is made after he has utilised observations p_1^1, \dots, p_1^{J-1} to refine his knowledge of w , the vector of parameters conditioning his assessment of the p.d.f. of selling prices in the market for commodity 1.

The $(J-1)$ th component ξ_{J-1}^{bJ} of the backward induction stopping rule ξ^{bJ} is defined as choosing the taking of observation p_1^J if and only if the consumer expects to attain a higher level of utility by so doing, ie.,

$$\xi_{J-1}^{bJ}(y_{J-1}) = S_{J-1}^{bJ} = \begin{cases} 0, & U(\delta(y_{J-1})) < E[U(\delta(y_J)) | y_{J-1}] \\ 1, & U(\delta(y_{J-1})) \geq E[U(\delta(y_J)) | y_{J-1}] \end{cases} \quad 2-3-5$$

By definition,

$$V_{J-1}^J(\xi_{J-1}^{bJ}, \delta) = \max\{U(\delta(y_{J-1})), E[U(\delta(y_J)) | y_{J-1}]\} \quad 2-3-6$$

$$= \max\{U(\delta(y_{J-1})), E[V_J^J(\xi_{J-1}^{bJ}, \delta) | y_{J-1}]\} \quad 2-3-7$$

by (2-3-4) so (2-3-5) may be rewritten as

$$\xi_{J-1}^{bJ}(y_{J-1}) = S_{J-1}^{bJ} = \begin{cases} 0, & U(\delta(y_{J-1})) \neq V_{J-1}^J(\xi_{J-1}^{bJ}, \delta) \\ 1, & U(\delta(y_{J-1})) = V_{J-1}^J(\xi_{J-1}^{bJ}, \delta) \end{cases} \quad 2-3-8$$

Continuing on in this fashion for $j = J-2, J-3, \dots$ shows that the j th component ξ_j^{bJ} of the backward induction stopping rule ξ^{bJ} is

$$\xi_j^{bJ}(y_j) = S_j^{bJ} = \begin{cases} 0, & U(\delta(y_j)) \neq V_j^J(\xi^{bJ}, \delta) \\ 1, & U(\delta(y_j)) = V_j^J(\xi^{bJ}, \delta) \end{cases} \quad 2-3-9$$

for all $j = 0, \dots, J-1$. The meaning of (2-3-9) is that, given the values of his previous observations p_1^1, \dots, p_1^j , if the consumer expects the greatest utility attainable at this or any subsequent stage of his search, $V_j^J(\xi^{bJ}, \delta)$, to be attainable now, then he should stop his search ($S_j^{bJ} = 1$). If the consumer expects the greatest utility attainable at any stage of his search to be attainable at some later stage of his search, then he should continue his search ($S_j^{bJ} = 0$). The recursion contained in (2-3-9) is a formal expression of Bellman's optimality principle.

Note that the first component ξ_0^{bJ} of ξ^{bJ} is the market entry decision component of the consumer's search rule, ie.

$$\xi_0^{bJ}(y_0) = S_0^{bJ} = \begin{cases} 0, & U(\delta(y_0)) \neq V_0^J(\xi^{bJ}, \delta) \text{ ie. } U(\delta(y_0)) < E[V_1^J(\xi^{bJ}, \delta)] \\ 1, & U(\delta(y_0)) = V_0^J(\xi^{bJ}, \delta) \text{ ie. } U(\delta(y_0)) \geq E[V_1^J(\xi^{bJ}, \delta)] \end{cases} \quad 2-3-10$$

Thus the consumer chooses to not enter the market for commodity 1 only if $U(\delta(y_0))$, the utility attainable from purchases of only commodities 2 to ℓ , exceeds $E[V_1^J(\xi^{bJ}, \delta)]$, the utility the consumer expects ex ante to be attainable at any stage of a search. Recall from Section 2-1 that the first component of the sequential stopping rules used by Nelson [34], Rothschild [43], Telser [52], Gastwirth [11], McCall [32] and Axell [1] is restricted to always committing the consumer to entering the market for commodity 1, ie.

$$\xi_0^{bJ}(y_0) = S_0^{bJ} \equiv 0 \quad 2-3-11$$

The stopping rule ξ^{bJ} used in this thesis makes no such market entry restriction. Note also that ξ^{bJ} is not a myopic stopping rule. In deciding whether or not to take observation p_1^j , for any $j \geq 1$, the

consumer considers the utility he expects to attain at *all* subsequent stages $j+1, \dots, J$ of his search. A myopic stopping rule merely considers the utility the consumer expects to attain at the very next stage of his search. The last section of this chapter explores the conditions under which the stopping rule ξ^{bJ} becomes myopic.

SECTION 2-4: THE OPTIMAL TERMINAL DECISION PROCEDURE FOR THE SEQUENTIAL SEARCH MODEL

Bellman's optimality principle implies that, when the searcher halts his search, he should choose the terminal action which gives the maximum attainable utility for the situation he finds himself to be in.

First of all, suppose the searcher chooses not to search. His terminal action set is $A_0 = \{a_0\}$ so that his only possible terminal action is a_0 . This is a formal way of saying that commodity 1 can only be purchased if at least one seller is contacted ie. some search must be undertaken if commodity 1 is to be purchased. The maximum level of utility attainable from allocating wealth M to only commodities 2 to ℓ is $U_0^*(0)$ where

$$U_0^*(0) = \max\{U \mid U = U(0, x_2, \dots, x_\ell), p_2 x_2 + \dots + p_\ell x_\ell \leq M\} \quad 2-4-1$$

Now suppose the consumer chooses to enter the market and chooses to stop searching after making $j \geq 1$ observations on p_1 . The financial cost of search incurred is $c(j)$, reducing the searcher's net wealth to

$$\bar{M}_j = M - c(j) \quad 2-4-2$$

His terminal action set is

$$A_j = \{a_0, a_1, a_2, \dots, a_j\} \quad 2-4-3$$

The set of transaction costs associated with A_j is

$$T_j = \{0, t_1, t_2, \dots, t_j\} \quad 2-4-4$$

If terminal action a_0 is chosen, the searcher exits from the market

without purchasing any of commodity 1 ($x_1^* = 0$) and incurs no transaction cost. If action a_i is chosen, for any $i = 1, \dots, j$, the searcher exits from the market by purchasing a unit of commodity 1 from the i th seller contacted ($x_1^* = 1$) and incurs a transaction cost of t_i . The total payment to the i th seller is $(p_1^i + t_i)$ leaving wealth $(\bar{M}_j - (p_1^i + t_i))$ to be allocated to commodities 2 to ℓ . The consumer's optimal terminal action is the action which maximises the utility attainable. To choose this action the consumer must compare the levels of utility attainable with each of the $(j+1)$ possible terminal actions.

The greatest utility attainable with action $a_0 \in A_j$ is $U_j^*(0)$ where

$$U_j^*(0) = \max\{U \mid U = U(0, x_2, \dots, x_\ell), p_2 x_2 + \dots + p_\ell x_\ell \leq \bar{M}_j\} \quad 2-4-5$$

The greatest utility attainable with action $a_i \in A_j$ for any $i=1, \dots, j$ is

$$U_{ij}^*(1) = \max\{U \mid U = U(1, x_2, \dots, x_\ell), p_2 x_2 + \dots + p_\ell x_\ell \leq \bar{M}_j - (p_1^i + t_i)\} \quad 2-4-6$$

The most preferred of actions a_1, \dots, a_j is the action which provides the highest $U_{ij}^*(1)$. This maximum will be denoted by $U_j^*(1)$ ie.

$$U_j^*(1) = \max_{1 \leq i \leq j} \{\max\{U \mid U = U(1, x_2, \dots, x_\ell), p_2 x_2 + \dots + p_\ell x_\ell \leq \bar{M}_j - (p_1^i + t_i)\}\} \quad 2-4-7$$

Combining (2-4-5) and (2-4-7) shows the highest utility attainable after $j = 1, 2, \dots$ observations on p_1 have been taken is

$$U_j^*(x) = \max\{\max\{U \mid U = U(x, x_2, \dots, x_\ell), p_2 x_2 + \dots + p_\ell x_\ell \leq \bar{M}_j - (p_1^i + t_i)x\}\} \quad 2-4-8$$

where $x = 0, 1$.

If $U_j^*(0) > U_j^*(1)$, then the highest possible level of utility is attained by exiting from the market without purchasing any of commodity 1. Hence,

$$\delta^*(y_j) = a_0 \text{ if } U_j^*(x) = U_j^*(0) > U_j^*(1) \quad 2-4-9$$

If $U_j^*(1) > U_j^*(0)$, then the highest possible level of utility is attained by purchasing a unit of commodity 1. Hence,

$$\delta^*(y_j) \in \{a_1, \dots, a_j\} \text{ if } U_j^*(0) < U_j^*(1) = U_j^*(x) \quad 2-4-10$$

With demand for commodity 1 fixed at either zero or one units it is

intuitive that the objective of the consumer is to minimise his expendi-

ture on commodity 1 through prices, transaction costs and search costs so as to have as much wealth as possible remaining for allocation to commodities 2 to ℓ . Intuitively, therefore, the seller offering a utility level of $U_j^*(1)$ to the searcher is the seller whose selling price and transaction cost combined is least.

Proposition 2-4-1:

Given that a sequential search halts after j observations on p_1 (for any $j = 0, 1, 2, \dots$), the optimal terminal decision procedure

$\delta^*(y_j) \in A_j$ is

$$\delta^*(y_j) = \begin{cases} a_0, & \text{if } U_j^*(0) > U_j^*(1) \text{ or if } j = 0 \\ a_i, & \text{if } j \geq 1 \text{ and } U_j^*(0) < U_j^*(1) \text{ and } (p_1^i + t_i) = (p_1 + t_j)^{\min} \end{cases}$$

Proof:

Given search has halted after $j \geq 1$ observations on p_1 , δ^* chooses the terminal action which provides the highest attainable level of utility $U_j^*(x)$. (2-4-9) establishes that

$$\delta^*(y_j) = a_0 \text{ if } U_j^*(x) = U_j^*(0) > U_j^*(1)$$

(2-4-10) establishes that

$$\delta^*(y_j) \in \{a_1, \dots, a_j\} \text{ if } U_j^*(x) = U_j^*(1) > U_j^*(0).$$

The utility attainable from taking action $a_i \in \{a_1, \dots, a_j\}$ is, from (2-4-6),

$$U_{ij}^*(1) = \max\{U | U = U(1, x_2, \dots, x_\ell), p_2 x_2 + \dots + p_\ell x_\ell \leq \bar{M}_j - (p_1^i + t_i)\}$$

The maximum $U_{ij}^*(1)$ attainable, $U_j^*(1)$, will be provided by the terminal action a_i which provides the largest feasible set of solutions x_2, \dots, x_ℓ defined by the wealth constraint

$$p_2 x_2 + \dots + p_\ell x_\ell \leq \bar{M}_j - (p_1^i + t_i) \quad 2-4-11$$

The set of solutions x_2, \dots, x_ℓ defined by (2-4-11) becomes larger as $(\bar{M}_j - (p_1^i + t_i))$ increases ie. as $(p_1^i + t_i)$ decreases. Hence

$$\delta^*(y_j) = a_i \text{ if } U_j^*(1) > U_j^*(0) \text{ and}$$

$$(p_1^i + t_i) = \min\{p_1^1 + t_1, \dots, p_1^j + t_j\} = (p_1 + t)_j^{\min} \quad 2-4-12$$

Q.E.D.

The inclusion of transaction costs into a search model with zero-one demand for commodity 1 is intuitive and not difficult. In Section 6-5 transaction costs are included in a generalised search model with continuously variable demand for all commodities 1 to ℓ . The resulting optimal terminal decision procedure provides an explanation of why transaction costs prevent small amounts of commodity 1 being purchased.

If the transaction costs associated with different sellers are all equal, the seller who represents the least expenditure on commodity 1 to the searcher will again be the seller who offers the lowest observed price.

Corollary 2-4-2:

Given sequential search halts after j observations on p_1 (for any $j = 0, 1, 2, \dots$), and given $t_i = t$, a constant, for all $i = 1, \dots, j$, the optimal terminal decision procedure $\delta^*(y_j) \in A_j$ is

$$\delta^*(y_j) = \begin{cases} a_0, & \text{if } U_j^*(0) > U_j^*(1) \text{ or if } j = 0 \\ a_k, & \text{if } j \geq 1 \text{ and } U_j^*(0) < U_j^*(1) \text{ and } p_{1j}^{\min} = p_1^k \end{cases}$$

Proof:

If $t_i = t$ for all $i = 1, \dots, j$, then

$$(p_1^k + t_k) = \min_{i=1, \dots, j} (p_1^i + t_i) \text{ if and only if } p_1^k = p_{1j}^{\min} \quad 2-4-13$$

The result follows immediately from Proposition 2-4-1. Q.E.D.

Note that even if all transaction costs are zero, the optimal terminal decision procedure does not degenerate to the terminal decision procedure used by Stigler, Rothschild, etc. where a unit of commodity 1

must always be purchased from the seller who offered the lowest price. δ^* still allows the option of buying zero units of commodity 1. This terminal action will be taken more frequently as transaction costs increase, resulting in a decline in the market demand for commodity 1. The option of buying zero units of commodity 1 also avoids the contradiction inherent in the simple rule (2-1-1) between a fixed demand hypothesis and an objective of minimisation of expenditure on commodity 1 (see Section 2-1).

Although expenditure minimisation is a term used throughout this section it should be remembered that δ^* does not minimise expenditure on commodity 1. The minimum possible expenditure on commodity 1 is zero i.e. $x_1^* = 0$ for $j = 0$. δ^* minimises expenditure on commodity 1 only to the extent that, when $x_1^* = 1$, δ^* selects the seller from whom a unit of commodity 1 can be purchased with least expenditure.

The next section shows that $\rho_s^{bJ} = (\xi^{bJ}, \delta^*)$ is an optimal sequential search rule, subject to conditions sufficient to ensure that no more than a finite number J of observations will be taken on p_1 .

SECTION 2-5: THE OPTIMALITY OF THE SEQUENTIAL SEARCH RULE

In this section the sequential search rule $\rho_s^{bJ} = (\xi^{bJ}, \delta^*)$ is shown to maximise ex ante expected utility from search and, therefore, to be optimal. The proof is initially subject to assumption (2-3-1) that only a finite number of observations on p_1 will ever be taken. It is then proved that searchers using the optimal search rule (ξ^{bJ}, δ^*) will always take only a finite number of observations on p_1 if financial search costs exist. This shows assumption (2-3-1) to be unnecessary, so that the optimality proof stands without the assumption.

By definition (2-2-19) and by assumption (2-3-1), $\rho_s^{bJ} = (\xi^{bJ}, \delta^*)$

is optimal if and only if

$$V_0^J(\xi^{bJ}, \delta^*) \geq V_0^J(\xi^J, \delta) \quad 2-5-1$$

for any other sequential search rule (ξ^J, δ) . $V_0^J(\xi^{bJ}, \delta^*)$ is the searcher's ex ante expected maximum attainable utility from using the search rule (ξ^{bJ}, δ^*) . This expectation must include the searcher's assessment of the relative likelihoods and magnitudes of different attainable levels of utility. For instance, a utility level of $\max\{U_k^*(0), U_k^*(1)\}$ can be enjoyed if and only if search halts after k observations on p_1 . Both the likelihood of search halting after k observations and the value of $U_k^*(1)$ depend upon p_1^1, \dots, p_1^k , the values of the k observations on p_1 . The vector ψ^{bJ} , defined by (1-4-21), is a particularly convenient form of notation to use to emphasise that the enjoyment of a utility level of $\max\{U_k^*(0), U_k^*(1)\}$ is conditional upon search halting after exactly k observations on p_1 . The k th component of ψ^{bJ} is, from (1-4-21),

$$\psi_k^{bJ}(y_k) = \psi_k^{bJ} = (1-S_0^{bJ}) \dots (1-S_{k-1}^{bJ}) S_k^{bJ} = \begin{cases} 0, & \text{otherwise} \\ 1, & \text{if search halts after} \\ & \text{exactly } k \text{ observations on } p_1 \end{cases} \quad 2-5-3$$

ξ_i^{bJ} is a function of p_1^1, \dots, p_1^i , for all $i = 1, \dots, k$, so ψ_k^{bJ} is also a function of p_1^1, \dots, p_1^k . A utility level of $\max\{U_k^*(0), U_k^*(1)\}$ is enjoyed if and only if $\psi_k^{bJ} = 1$. From (2-5-3), if $\psi_k^{bJ} = 1$, $\psi_i^{bJ} = 0$ for all $i = 0, \dots, J$, $i \neq k$. Hence, if search halts after k observations on p_1 , the utility level enjoyed is

$$\max\{U_k^*(0), U_k^*(1)\} = \psi_0^{bJ} \cdot U_0^*(0) + \sum_{j=1}^J \psi_j^{bJ} \cdot \max\{U_j^*(0), U_j^*(1)\} \quad 2-5-4$$

The searcher's ex ante expected attainable utility $V_0^J(\xi^{bJ}, \delta^*)$ is his expectation w.r.t. selling prices on the magnitude and relative likelihoods of the $(J+1)$ possible outcomes from search $U_0^*(0)$,

$\max\{U_1^*(0), U_1^*(1)\}, \dots, \max\{U_J^*(0), U_J^*(1)\}$. Intuitively, therefore,

$$V_0^J(\xi^{bJ}, \delta^*) = E[\psi_0^{bJ} \cdot U_0^*(0) + \sum_{j=1}^J \psi_j^{bJ} \cdot \max\{U_j^*(0), U_j^*(1)\}] \quad 2-5-5$$

This intuition is formalised in Lemma 2-5-1 which is then combined with Lemma 2-5-2 to prove the optimality of the search rule (ξ^{bJ}, δ^*) in Proposition 2-5-3.

Lemma 2-5-1:

$$V_0^J(\xi^{bJ}, \delta^*) = \psi_0^{bJ} \cdot U_0^*(0) + \sum_{j=1}^J \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_j^{bJ} \cdot \max\{U_j^*(0), U_j^*(1)\} \cdot f_g(p_1^1, \dots, p_1^j) dp_1^1 \dots dp_1^j$$

Proof:

From (2-3-10),

$$V_0^J(\xi^{bJ}, \delta^*) = \max\{U_0^*(0), E[V_1^J(\xi^{bJ}, \delta^*)]\} \quad 2-5-6$$

$$= S_0^{bJ} \cdot U_0^*(0) + (1-S_0^{bJ}) \cdot E[V_1^J(\xi^{bJ}, \delta^*)] \quad 2-5-7$$

$$V_1^J(\xi^{bJ}, \delta^*) = \max\{U_1^*(0), U_1^*(1), E[V_2^J(\xi^{bJ}, \delta^*) | y_1]\} \quad 2-5-8$$

From (2-5-7), (2-5-8) and (2-3-9),

$$V_0^J(\xi^{bJ}, \delta^*) = S_0^{bJ} \cdot U_0^*(0) + (1-S_0^{bJ}) \cdot E[S_1^{bJ} \cdot \max\{U_1^*(0), U_1^*(1)\} + (1-S_1^{bJ}) \cdot E[V_2^J(\xi^{bJ}, \delta^*) | y_1]] \quad 2-5-9$$

and so on up to

$$V_0^J(\xi^{bJ}, \delta^*) = S_0^{bJ} \cdot U_0^*(0) + (1-S_0^{bJ}) \cdot E[S_1^{bJ} \cdot \max\{U_1^*(0), U_1^*(1)\} + (1-S_1^{bJ}) \cdot E[\dots \\ \dots + (1-S_{J-1}^{bJ}) \cdot E[S_J^{bJ} \cdot \max\{U_J^*(0), U_J^*(1)\} | y_{J-1} \dots | y_1]]] \quad 2-5-10$$

Recall that ξ_i^{bJ} is a function of p_1^1, \dots, p_1^i for all $i = 1, \dots, J$.

$$V_0^J(\xi^{bJ}, \delta^*) = S_0^{bJ} \cdot U_0^*(0) + (1-S_0^{bJ}) \int_{p_1^L}^{p_1^U} S_1^{bJ} \cdot \max\{U_1^*(0), U_1^*(1)\} \cdot f_g(p_1^1) dp_1^1 + \quad 2-5-11$$

$$\dots + (1-S_0^{bJ}) \int_{p_1^L}^{p_1^U} (1-S_1^{bJ}) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} S_J^{bJ} \cdot \max\{U_J^*(0), U_J^*(1)\} \cdot f_g(p_1^1, \dots, p_1^J) dp_1^1 \dots dp_1^J$$

$$= S_0^{bJ} \cdot U_0^*(0) \quad 2-5-12$$

$$\begin{aligned}
& + \sum_{j=1}^J \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (1-s_0^{bJ}) \dots (1-s_{j-1}^{bJ}) s_j^{bJ} \cdot \max\{U_j^*(0), U_j^*(1)\} \cdot f_g(p_1^1, \dots, p_1^j) dp_1^1 \dots dp_1^j \\
& = \psi_0^{bJ} \cdot U_0^*(0) + \sum_{j=1}^J \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_j^{bJ} \cdot \max\{U_j^*(0), U_j^*(1)\} \cdot f_g(p_1^1, \dots, p_1^j) dp_1^1 \dots dp_1^j
\end{aligned}$$

2-5-13

Q.E.D.

Lemma 2-5-2 proves a general result commonly used in the statistics of sequential decision making. The result characterises a function p^* which maximises the integral of a linear combination of two continuous functions a and b . It will be shown in Proposition 2-5-3 that each of the components ξ_j^{bJ} of the backward induction stopping rule ξ^{bJ} have the same form as p^* and consequently maximise an integral which is a linear combination of expected utilities. This property is used to prove ξ^{bJ} is an optimal sequential stopping rule.

Lemma 2-5-2:

Given two continuous functions $a(x)$ and $b(x)$, the function $p(x)$, $0 \leq p(x) \leq 1$, which maximises the integral

$$\int_X (p(x) \cdot a(x) + (1-p(x)) \cdot b(x)) dx$$

is

$$p^*(x) = \begin{cases} 0, & a(x) < b(x) \\ 1, & a(x) \geq b(x) \end{cases}$$

Proof:

The above statement of $p^*(x)$ is equivalent to

$$p^*(x) = \begin{cases} 0, & a(x) < b(x) \\ 1, & a(x) \geq b(x) \end{cases}$$

2-5-14

$$\begin{aligned} \text{Hence } \int_{\chi} (p^*(x) \cdot a(x) + (1-p^*(x)) \cdot b(x)) dx \\ = \int_{\chi} \max\{a(x), b(x)\} dx \end{aligned} \quad 2-5-15$$

$$\text{Since } \max\{a(x), b(x)\} \geq p(x)a(x) + (1-p(x))b(x) \quad 2-5-16$$

for any $0 \leq p(x) \leq 1$ and for all $x \in \chi$,

$$\int_{\chi} (p^*(x)a(x) + (1-p^*(x))b(x)) dx \geq \int_{\chi} (p(x)a(x) + (1-p(x))b(x)) dx \quad 2-5-17$$

for any $p(x)$, $0 \leq p(x) \leq 1$.

Q.E.D.

It will assist the reader's understanding of the proof of the optimality of the sequential search rule (ξ_j^{bJ}, δ^*) if he compares the form of $p^*(x)$ to the form of ξ_j^{bJ} in (2-3-9). Recall that

$$V_j^J(\xi_j^{bJ}, \delta^*) = \max\{U_j^*(0), U_j^*(1), E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]\} \quad 2-5-18$$

$$= \max\{\max\{U_j^*(0), U_j^*(1)\}, E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]\} \quad 2-5-19$$

From (2-3-9) and (2-5-19), ξ_j^{bJ} can be written as

$$\xi_j^{bJ}(y_j) = S_j^{bJ} = \begin{cases} 0, & \max\{U_j^*(0), U_j^*(1)\} \neq \max\{\max\{U_j^*(0), U_j^*(1)\}, \\ & E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]\} \\ 1, & \max\{U_j^*(0), U_j^*(1)\} = \max\{\max\{U_j^*(0), U_j^*(1)\}, \\ & E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]\} \end{cases} \quad 2-5-20$$

Comparison of (2-5-20) to the statement of Lemma 2-5-2 will show the similarity of the forms of p^* and ξ_j^{bJ} . p^* takes a value of zero if the maximum of the functions a and b for some x is the latter function b .

ξ_j^{bJ} takes a value of zero if the maximum of the functions $\max\{U_j^*(0), U_j^*(1)\}$ and $E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]$ for some p_1 is the latter function $E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]$. p^* takes a value of unity if the maximum of the functions a and b for some x is the former function a . ξ_j^{bJ} takes a value of unity if the

maximum of the functions $\max\{U_j^*(0), U_j^*(1)\}$ and $E[V_{j+1}^J(\xi_j^{bJ}, \delta^*) | y_j]$ for some p_1 is the former function $\max\{U_j^*(0), U_j^*(1)\}$. The similarity of the forms of p^* and ξ_j^{bJ} allows Lemma 2-5-2 to be used in Proposition 2-5-3 which proves the backward induction stopping rule ξ^{bJ} is an optimal sequential stopping rule.

Proposition 2-5-3:

ρ^{bJ} is an optimal sequential search rule provided no more than J observations will be taken on p_1 .

Proof:

ρ^{bJ} is an optimal sequential search rule if and only if (2-5-1) is satisfied. The first step in the proof is to show that $V_0^J(\xi^{bJ}, \delta) \geq V_0^J(\xi^J, \delta)$ for any sequential stopping rule ξ^J and any terminal decision procedure δ . The second step is to show that $V_0^J(\xi^{bJ}, \delta^*) \geq V_0^J(\xi^{bJ}, \delta)$ for any terminal decision procedure δ .

Consider any sequential search rule $\rho_s^J = (\xi^J, \delta)$. By repeating the logic of Lemma 2-5-1, $V_0^J(\xi^J, \delta)$, the ex ante expected utility from the use of search rule ρ_s^J , can be written in a form analogous to (2-5-10) as

$$\begin{aligned} V_0^J(\xi^J, \delta) = & S_0^J U(\delta(y_0)) + (1-S_0^J) E[S_1^J U(\delta(y_1)) + (1-S_1^J) E[S_2^J U(\delta(y_2)) + \\ & \dots E[S_{J-1}^J U(\delta(y_{J-1})) + (1-S_{J-1}^J) E[S_J^J U(\delta(y_J)) | y_{J-1}] | y_{J-2}] \\ & \dots | y_2] | y_1]] \end{aligned} \quad 2-5-21$$

Note that, since $A_0 = \{a_0\}$,

$$U(\delta(y_0)) = U^*(0) = U(\delta^*(y_0)) \quad 2-5-22$$

for any terminal decision procedure δ and that

$$\xi_J^J(y_J) = S_J^J \equiv 1 \equiv S_J^{bJ} = \xi_J^{bJ}(y_J) \quad 2-5-23$$

for any sequential stopping rule ξ^J . Hence (2-5-21) can be written as

$$\begin{aligned}
V_0^J(\xi^J, \delta) &= S_0^J U_0^*(0) + (1-S_0^J) \int_{p_1^L}^{p_1^U} [S_1^J U(\delta(y_1)) + (1-S_1^J) \int_{p_1^L}^{p_1^U} [S_2^J U(\delta(y_2)) + \\
&\quad \dots + \int_{p_1^L}^{p_1^U} [S_{J-1}^J U(\delta(y_{J-1})) + (1-S_{J-1}^J) \int_{p_1^L}^{p_1^U} U(\delta(y_J)) f_g(p_1^J | y_{J-1}) dp_1^J] \\
&\quad f_g(p_1^{J-1} | y_{J-2}) dp_1^{J-1}] \dots] f_g(p_1^2 | y_1) dp_1^2] f_g(p_1^1) dp_1^1
\end{aligned} \tag{2-5-24}$$

The j th individual integral, $j = 0, \dots, J-1$, of the nest of integrals (2-5-24) is

$$\int_{p_1^L}^{p_1^U} [S_j^J U(\delta(y_j)) + (1-S_j^J) \int_{p_1^L}^{p_1^U} [\dots] f_g(p_1^{j+1} | y_j) dp_1^{j+1}] f_g(p_1^j | y_{j-1}) dp_1^j \tag{2-5-25}$$

which is an integral of the type considered in Lemma 2-5-2. Applying Lemma 2-5-2 shows the function $\xi_j^J(y_j)$ which maximises (2-5-25) is $\xi_j^{J*}(y_j)$ where

$$\xi_j^{J*}(y_j) = S_j^{J*} = \begin{cases} 0, & U(\delta(y_j)) \neq \max\{U(\delta(y_j)), \int_{p_1^L}^{p_1^U} [\dots] f_g(p_1^{j+1} | y_j) dp_1^{j+1}\} \\ 1, & U(\delta(y_j)) = \max\{U(\delta(y_j)), \int_{p_1^L}^{p_1^U} [\dots] f_g(p_1^{j+1} | y_j) dp_1^{j+1}\} \end{cases} \tag{2-5-26}$$

which, by the definition of $V_j^J(\xi^J, \delta)$, is equivalent to

$$\xi_j^{J*}(y_j) = S_j^{J*} = \begin{cases} 0, & U(\delta(y_j)) \neq V_j^J(\xi^{J*}, \delta) \\ 1, & U(\delta(y_j)) = V_j^J(\xi^{J*}, \delta) \end{cases} \tag{2-5-27}$$

Comparison of (2-5-27) and (2-3-9) shows $\xi_j^{J*} \equiv \xi_j^{bJ}$ and this is true for all $j = 0, \dots, J$. Hence $\xi^{J*} \equiv \xi^{bJ}$ and therefore

$V_0^J(\xi^{bJ}, \delta) \geq V_0^J(\xi^J, \delta)$ for any sequential search rule ξ^J and any terminal decision procedure δ . 2-5-28

It remains to show that $V_0^J(\xi^{bJ}, \delta^*) \geq V_0^J(\xi^{bJ}, \delta)$. Recall from Section 2-4 that

$$\delta^*(y_j) = \max\{U_j^*(0), U_j^*(1)\} \geq \max\{U_j^*(0), U_{ij}^*(1)\} \quad 2-5-29$$

for any terminal action $a_i \in A_j$. Hence,

$$U(\delta^*(y_j)) \geq U(\delta(y_j)) \text{ for any terminal decision procedure } \delta. \quad 2-5-30$$

Note, by comparison of (2-5-25) to (2-5-15), that

$$\begin{aligned} & \int_{P_1^L}^{P_1^U} [S_j^{bJ} U(\delta(y_j)) + (1-S_j^{bJ}) \int_{P_1^L}^{P_1^U} [\dots] f_g(p_1^{j+1}|y_j) dp_1^{j+1}] f_g(p_1^j|y_{j-1}) dp_1^j \\ &= \int_{P_1^L}^{P_1^U} \max[U(\delta(y_j)), \int_{P_1^L}^{P_1^U} [\dots] f_g(p_1^{j+1}|y_j) dp_1^{j+1}] f_g(p_1^j|y_{j-1}) dp_1^j \end{aligned} \quad 2-5-31$$

From (2-5-24), $V_0^J(\xi^{bJ}, \delta)$ may therefore be written as

$$\begin{aligned} V_0^J(\xi^{bJ}, \delta) &= \max[U_0^*(0), \int_{P_1^L}^{P_1^U} \max[U(\delta(y_1)), \int_{P_1^L}^{P_1^U} \max[U(\delta(y_2)), \dots, \\ & \quad \int_{P_1^L}^{P_1^U} \max[U(\delta(y_{J-1})), \int_{P_1^L}^{P_1^U} U(\delta(y_J)) f_g(p_1^J|y_{J-1}) dp_1^J] f_g(p_1^{J-1}|y_{J-2}) dp_1^{J-1}] \\ & \quad \dots] f_g(p_1^2|y_1) dp_1^2] f_g(p_1^1) dp_1^1] \end{aligned} \quad 2-5-32$$

$$\begin{aligned} &\leq \max[U_0^*(0), \int_{P_1^L}^{P_1^U} \max[\max\{U_1^*(0), U_1^*(1)\}, \int_{P_1^L}^{P_1^U} \max[\max\{U_2^*(0), U_2^*(1)\}, \dots, \\ & \quad \int_{P_1^L}^{P_1^U} \max[\max\{U_{J-1}^*(0), U_{J-1}^*(1)\}, \int_{P_1^L}^{P_1^U} \max\{U_J^*(0), U_J^*(1)\} f_g(p_1^J|y_{J-1}) dp_1^J] \\ & \quad f_g(p_1^{J-1}|y_{J-2}) dp_1^{J-1}] \dots] f_g(p_1^2|y_1) dp_1^2] f_g(p_1^1) dp_1^1] \end{aligned} \quad 2-5-33$$

$$\begin{aligned}
&= \max[U_0^*(0), \int_{p_1^L}^{p_1^U} \max[U(\delta^*(y_1)), \int_{p_1^L}^{p_1^U} \max[U(\delta^*(y_2)), \dots, \\
&\quad \int_{p_1^L}^{p_1^U} \max[U(\delta^*(y_{J-1})), \int_{p_1^L}^{p_1^U} U(\delta^*(y_J)) f_g(p_1^J | y_{J-1}) dp_1^J] f_g(p_1^{J-1} | y_{J-2}) dp_1^{J-1}] \\
&\quad \dots] f_g(p_1^2 | y_1) dp_1^2] f_g(p_1^1) dp_1^1] \\
&= V_0^J(\xi^{bJ}, \delta^*) \tag{2-5-34} \\
&= V_0^J(\xi^{bJ}, \delta^*) \tag{2-5-35}
\end{aligned}$$

Combining (2-5-35) and (2-5-28) shows

$$V_0^J(\xi^{bJ}, \delta^*) \geq V_0^J(\xi^J, \delta) \text{ for any sequential search rule } \rho^J = (\xi^J, \delta).$$

Q.E.D.

The reader may wish to note that applying results such as Lemma 2-5-2 in the manner used in Proposition 2-5-3 is a quite general technique applicable to many problems in this area of research.

Proposition 2-5-3 is still subject to assumption (2-3-1) that there exists a finite upper bound J on the number of observations on p_1 taken by any searcher using the optimal search rule (ξ^{bJ}, δ^*) . However, this assumption is very weak for two reasons. First, the real world contains only a finite number of sellers. Second, the existence of search costs means that the searcher will always choose to contact only a finite number of sellers. This means that in reality this assumption is redundant. The searcher's initial wealth M is finite. If there is a finite cost associated with each observation on p_1 , only a finite number of observations can be taken before search costs reduce the searcher's net wealth to zero and so reduce the attainable utility to the minimum. The searcher will always prefer to halt search before this occurs so a finite bound J will always exist.

Lemma 2-5-4:

In the presence of financial search costs a searcher using the

optimal search rule (ξ^{bJ}, δ^*) will always make only a finite number of observations on p_1 .

Proof:

The total financial cost of taking j observations on p_1 is $c(j)$. Let J' be the smallest integer such that

$$c(J') \geq M \qquad \qquad \qquad 2-5-36$$

ie. the searcher can afford to take at most J' observations on p_1 . J' is therefore an upper bound on the number of observations a searcher will take on p_1 . All marginal financial search costs are assumed to be positive and finite by (2-2-14) and M is finite. Therefore J' is finite.

Q.E.D.

Lemma 2-5-4 depends crucially on assumption (2-2-14) that any observation exacts a financial cost from the searcher. This assumption is not always realistic since, in some searches, the searcher may incur no financial costs in making observations on p_1 . For example, sellers do not charge potential buyers for reading their advertisements. In this search model, however, zero financial search costs imply the finite bound J does not exist (a Type 1 problem, see Section 1-4). This is because psychic costs of search are ignored (assumption (2-2-14)) when, as Stigler [50] pointed out, they are a major deterrent to search. In Chapter 5 this deficiency is removed by presenting the formalised Stigler search model containing both financial and psychic costs of search.

Assumption (2-2-14) is also unnecessarily strong in that it requires a positive marginal financial cost on each observation on p_1 . However, there are searches in which a small payment is made to a searcher for at least some of his observations on p_1 . For example, free food and drink is sometimes offered to potential buyers to entice

them to a sales promotion. All that is required for Lemma 2-5-4 to be valid is that accumulated financial search costs eventually exceed the searcher's initial wealth, even if some of the earlier marginal costs are negative (payments) ie. there must be some finite J such that

$$c(J') \geq M \text{ for all } J' \geq J \quad 2-5-37$$

The derivation of the sequential stopping rule ξ^{bJ} in Section 2-3 relies upon an assumption of a finite value for J . In Proposition 2-5-5 it is proved that the optimality of the search rule (ξ^{bJ}, δ^*) does not depend upon the assumption.

Proposition 2-5-5:

The sequential search rule $\rho_s^{bJ} = (\xi^{bJ}, \delta^*)$ is always optimal for consumer search problems characterised by (2-2-1), (2-2-2), (2-2-13), (2-2-14), (2-2-15), (2-2-16) and (2-2-18).

Proof:

The search rule $\rho_s^{bJ} = (\xi^{bJ}, \delta^*)$ was proved optimal for consumer search problems characterised by (2-2-1), (2-2-2), (2-2-13), (2-2-15), (2-2-16) and (2-2-18) in which at most J observations on p_1 will be made. Lemma 2-5-4 proved (2-2-14) was sufficient to guarantee that no more than J observations on p_1 will ever be made in consumer search problems with the above characteristics. Hence $\rho_s^{bJ} = (\xi^{bJ}, \delta^*)$ will always be optimal for these problems.

Q.E.D.

From this point on the optimal sequential search rule will be denoted by

$$\rho_s^* = (\xi^*, \delta^*) \equiv (\xi^b, \delta^*) \quad 2-5-38$$

Both components of ρ_s^* , ξ^* and δ^* , are considerably more general and flexible than the stopping rule and terminal decision procedure implied

in the simple Rothschild, Telser, Gastwirth, Axell search rule given in (2-1-1). Note that with the change in notation in (2-5-38), (2-5-3) becomes

$$\psi_j^*(y_j) = \psi_j^* = (1-S_0^*) \dots (1-S_{j-1}^*) S_j^* \quad 2-5-39$$

and the statement of Lemma 2-5-1 becomes

$$V_0(\xi^*, \delta^*) = \psi_0^* U_0^*(0) + \sum_{j=1}^{\infty} \int_{P_1^L}^{P_1^U} \dots \int_{P_1^L}^{P_1^U} \psi_j^* \max\{U_j^*(0), U_j^*(1)\} \cdot f_g(p_1^1, \dots, p_1^j) dp_1^1 \dots dp_1^j \quad 2-5-40$$

The upper bound of J is replaced by ∞ in (2-5-13) since Lemma 2-5-4 guarantees $\psi_j^* = 0$ for all $j > J$.

The next section examines the sequential search rule which provides the lowest ex ante expected total expenditure on the purchase of a unit of commodity 1.

SECTION 2-6: THE SEQUENTIAL SEARCH RULE WHICH MINIMISES EX ANTE EXPECTED TOTAL EXPENDITURE ON THE PURCHASE OF A UNIT OF COMMODITY 1

$\hat{\rho}_s = (\hat{\xi}, \hat{\delta})$ is defined by (2-2-22) as being a sequential search rule for which the consumer's ex ante expectation of his total expenditure on commodity 1, $W_0(\xi, \delta)$, is a minimum with respect to his choice of sequential search rule $\rho_s = (\xi, \delta)$. $\hat{\rho}_s$ is subject to the condition that the consumer's demand for commodity 1 is $x_1^* \equiv 1$. A condition necessary for $x_1^* \equiv 1$ is that the consumer's market entry decision must always be to enter the market for commodity 1. Therefore, the first component $\hat{\xi}_0$ of the sequential stopping rule component $\hat{\xi}$ of $\hat{\rho}_s$ is restricted to being such that

$$\hat{\xi}_0(y_0) = \hat{S}_0 \equiv 0 \quad 2-6-1$$

The restriction $x_1^* \equiv 1$ is essential to the objective of minimising total expenditure on commodity 1 because it removes terminal action a_0 from the

consumer's terminal action space. If $x_1^* = 0$ (ie. terminal action a_0) is permissible with this objective then minimising total expenditure on commodity 1 requires that the terminal decision procedure $\hat{\delta}$ chooses terminal action a_0 always, ie. $x_1^* = 0$ always. This is because, by choosing a_0 , the consumer is inflicted only with his expenditure on search costs and avoids expenditure on purchase and transaction costs. Permitting $x_1^* = 0$ with an objective of minimising total expenditure on commodity 1 therefore causes the consumer search problem to vanish.

The techniques used in the analysis of Sections 2-3, 2-4 and 2-5 can be used to show that $\hat{\rho}_s$ has the following form.

$$\hat{\rho}_s = (\hat{\xi}, \hat{\delta}) \text{ where (i) } \hat{\xi} = (\hat{\xi}_0, \hat{\xi}_1, \dots, \hat{\xi}_j, \dots) \text{ where}$$

$$\hat{\xi}_0(y_0) = \hat{S}_0 \equiv 0 \text{ and where}$$

$$\hat{\xi}_j(y_j) = \hat{S}_j = \begin{cases} 0, & TE_j \neq W_j(\hat{\xi}, \hat{\delta}) \\ 1, & TE_j = W_j(\hat{\xi}, \hat{\delta}) \end{cases} \text{ for all } j \geq 1 \quad 2-6-2$$

The interpretation of (2-6-2) is as follows. By definition, $W_j(\hat{\xi}, \hat{\delta})$ is the minimum total expenditure that the consumer expects, after taking $j \geq 1$ observations on p_1 according to search rule $\hat{\rho}_s$, to be required for the purchase of a unit of commodity 1 at this or any subsequent stage of his search. If the consumer expects TE_j , the total expenditure required now, after j observations on p_1 , to be the minimum $W_j(\hat{\xi}, \hat{\delta})$, then he should choose to stop his search ($\hat{S}_j = 1$). If the consumer expects a minimum total expenditure $W_j(\hat{\xi}, \hat{\delta})$, lower than TE_j , to be attainable at some subsequent stage of his search then he should choose to continue his search ($\hat{S}_j = 0$).

(ii) Total expenditure on the purchase of a unit of commodity 1, after j observations on p_1 , is, if terminal action $a_k \in A_j$ is taken,

$$TE_j = p_1^k + t_k + c(j) \quad 2-6-3$$

$c(j)$ is the financial cost of taking observations p_1^1, \dots, p_1^j and is therefore independent of the terminal action chosen. Hence, the terminal decision procedure minimising total expenditure on commodity 1 subject to $x_1^* \equiv 1$ is $\hat{\delta}$ where

$$\hat{\delta}(y_j) = a_i \text{ if and only if } p_1^i + t_i = (p_1 + t)_j^{\min} \quad 2-6-4$$

The minimum total expenditure achieved, after j observations on p_1 , by using $\hat{\delta}$ will be denoted by $TE_j(\hat{\delta})$, ie.

$$TE_j(\hat{\delta}) = (p_1 + t)_j^{\min} + c(j) \quad 2-6-5$$

The reasoning of the proof of Proposition 2-5-3 can be adapted from an objective of maximising expected utility to an objective of minimising the expected total expenditure needed for the purchase of a unit of commodity 1. This allows the ex ante expectation of this expenditure, $W_0(\hat{\xi}, \hat{\delta})$, to be expressed in a form analogous to (2-5-34)

as

$$\begin{aligned} W_0(\hat{\xi}, \hat{\delta}) = & \int_{p_1^L}^{p_1^U} \min[TE_1(\hat{\delta})], \int_{p_1^L}^{p_1^U} \min[TE_2(\hat{\delta})], \int_{p_1^L}^{p_1^U} \min[TE_3(\hat{\delta})], \dots, \\ & \int_{p_1^L}^{p_1^U} \min[TE_j(\hat{\delta})], \int_{p_1^L}^{p_1^U} \min[TE_{j+1}(\hat{\delta}), \dots] f_g(p_1^{j+1} | y_j) dp_1^{j+1}] f_g(p_1^j | y_{j-1}) dp_1^j] \\ & \dots] f_g(p_1^3 | y_2) dp_1^3] f_g(p_1^2 | y_1) dp_1^2] f_g(p_1^1) dp_1^1 \end{aligned} \quad 2-6-6$$

The next section examines conditions under which one search rule simultaneously satisfies (2-6-6) and (2-5-34) ie. simultaneously minimises the consumer's ex ante expectation of the total expenditure required to purchase a unit of commodity 1 and maximises the consumer's ex ante expected utility.

SECTION 2-7: MYOPIC SEARCH RULES AND THE USE OF EXPENDITURE MINIMISATION AS A PROXY FOR UTILITY MAXIMISATION

In Section 1-3 it is stated that consumer search theorists commonly restrict the utility functions admissible to their problems to those functions for which utility maximisation is coincident with minimisation of expenditure on commodity 1. It is also stated that this restriction is usually accompanied by the restriction that utility functions must be such that demand for commodity 1 is fixed for every price-income situation possible from search. In Section 1-4 the severity of these restrictions is demonstrated for what are called Types 2 and 3 search problems. It is shown there that only utility functions with a certain degree and type of non-smoothness are admissible for these problems. In Section 2-1 a simple example is given demonstrating that an unqualified objective of minimising expenditure on commodity 1 can contradict the fixed demand hypothesis. Typically the users of the expenditure minimisation proxy have utilised its statistical simplicity but have had insufficient regard, if any, to the limits within which the proxy can be used. The purpose of this section is to demonstrate the severity of conditions which are necessary and sufficient to restrict the model developed in this paper to the myopic expenditure minimising models presented by Rothschild [43], Axell [1], Telser [52], and Gastwirth [11]. First to be examined are the conditions under which minimisation of the consumer's ex ante expected total expenditure on commodity 1 is equivalent to maximisation of the consumer's ex ante expected utility. Second to be explored are the conditions under which the consumer's optimal search rule is myopic.

The Expenditure Minimisation Proxy for Utility Maximisation:

The expenditure minimisation objective is expressed in terms of

prices and wealth. To ease comparison of this with the utility maximisation objective $V_0(\xi^*, \delta^*)$ will also be expressed in terms of prices and wealth. This is done by expressing (2-5-34) in terms of the consumer's indirect utility function I .

If, after $j \geq 0$ observations on p_1 , $\delta^*(y_j) = a_0$ then the consumer's demand for commodity 1 is zero units and his demands for commodities 1 to ℓ are

$$x_1^* = 0 \quad 2-7-1$$

$$x_i^* = x_i(p_2, \dots, p_\ell, \bar{M}_j) \text{ for all } i = 2, \dots, \ell \quad 2-7-2$$

The indirect utility derived from taking action a_0 after $j \geq 0$ observations have been taken on p_1 is therefore

$$U_j^*(0) = U(0, x_2(p_2, \dots, p_\ell, \bar{M}_j), \dots, x_\ell(p_2, \dots, p_\ell, \bar{M}_j)) \quad 2-7-3$$

$$= I_0(p_2, \dots, p_\ell, \bar{M}_j) \quad 2-7-4$$

If, after $j \geq 1$ observations on p_1 , $\delta^*(y_j) = a_i \neq a_0$ then the consumer's demand for commodity 1 is one unit and his demands for commodities 1 to ℓ are

$$x_1^* = 1 \quad 2-7-5$$

$$x_i^* = x_i(p_2, \dots, p_\ell, \bar{M}_j - (p_1 + t)_j^{\min}) \text{ for all } i = 2, \dots, \ell \quad 2-7-6$$

The indirect utility derived from taking action a_i after $j \geq 1$ observations have been taken on p_1 is therefore

$$U_j^*(1) = U(1, x_2(p_2, \dots, p_\ell, \bar{M}_j - (p_1 + t)_j^{\min}), \dots, x_\ell(p_2, \dots, p_\ell, \bar{M}_j - (p_1 + t)_j^{\min})) \quad 2-7-7$$

$$= I_1(p_2, \dots, p_\ell, \bar{M}_j - (p_1 + t)_j^{\min}) \quad 2-7-8$$

p_2, \dots, p_ℓ are fixed by assumption (1-4-1) so, for notational convenience, let

$$U_j^*(0) = I_0(p_2, \dots, p_\ell, \bar{M}_j) = \tilde{I}_0(\bar{M}_j) \quad 2-7-9$$

$$U_j^*(1) = I_1(p_2, \dots, p_\ell, \bar{M}_j - (p_1 + t)_j^{\min}) = \tilde{I}_1(\bar{M}_j - (p_1 + t)_j^{\min}) \quad 2-7-10$$

Substituting (2-7-9) and (2-7-10) into (2-5-34) shows $V_0(\xi^*, \delta^*)$ can be

consumer by demanding one unit of commodity 1. It has already been noted in Section 2-1 that, in the previous search models [43], [1], [52],[11], there is no guarantee that $x_1^* = 1$ will always be feasible. The source of the contradiction between the fixed demand condition $x_1^* \equiv 1$ and the myopic sequential stopping rule (2-1-1) is the next condition.

(iii) \tilde{I}_1 is linear with respect to $(\bar{M}_j - (p_1+t)_j^{\min})$ for all $j \geq 1$ i.e. utility is linear with respect to the net wealth the consumer is able to retain for allocation to purchases of commodities 2 to ℓ . This condition implies that the consumer is risk neutral with respect to expenditure on commodity 1 and it is this risk neutrality which permits the consumer to take a gamble which has the non-feasibility of $x_1^* = 1$ as a possible outcome. If a measure of risk neutrality is imposed on the consumer's utility function sufficient to prevent this gamble being taken, then \tilde{I}_1 is no longer linear with respect to $(\bar{M}_j - (p_1+t)_j^{\min})$ and expenditure minimisation is no longer a valid proxy for utility maximisation. Consequently some influence other than the consumer's preferences must be assumed present to ensure the fourth condition.

(iv) $x_1^* = 1$ is always feasible for any $j \geq 1$.

Definition:

$\Omega = \{U | U(x_1, \dots, x_\ell) \text{ is such that } \tilde{I}_0(M) < E[V_1(\xi^*, \delta^*)] \text{ and}$

$\tilde{I}_0(\bar{M}_j) < \tilde{I}_1(\bar{M}_j - (p_1+t)_j^{\min}) \text{ for all } j \geq 1\}$ 2-7-14

Proposition 2-7-1:

Minimisation of ex ante expected total expenditure on purchase of a unit of commodity 1 is a valid proxy for maximisation of ex ante expected utility if and only if

(i) $U \in \Omega$

(ii) $x_1^* = 1$ is always feasible

(iii) \tilde{I}_1 is linear with respect to $(\bar{M}_j - (p_1+t)_j^{\min})$ for all $j \geq 1$.

Proof:

(a) Sufficiency: $U \in \Omega$. Therefore (2-7-11) can be written as

$$V_0(\xi^*, \delta^*) = \int_{p_1^L}^{p_1^U} \max[\tilde{I}_1(\bar{M}_1 - (p_1+t)_1^{\min})], \int_{p_1^L}^{p_1^U} \max[\tilde{I}_1(\bar{M}_2 - (p_1+t)_2^{\min}), \dots, \\ \int_{p_1^L}^{p_1^U} \max[\tilde{I}_1(\bar{M}_j - (p_1+t)_j^{\min})], \int_{p_1^L}^{p_1^U} \max[\tilde{I}_1(\bar{M}_{j+1} - (p_1+t)_{j+1}^{\min}), \dots] \\ f_g(p_1^{j+1} | y_j) dp_1^{j+1}] f_g(p_1^j | y_{j-1}) dp_1^j] \dots] f_g(p_1^2 | y_1) dp_1^2] f_g(p_1^1) dp_1^1 \quad 2-7-15$$

$\bar{M}_j = M - c(j)$ by definition (1-4-18) so

$$\bar{M}_j - (p_1+t)_j^{\min} = M - ((p_1+t)_j^{\min} + c(j)) = M - TE_j(\delta^*) \quad 2-7-16$$

by Proposition 2-4-1 and conditions (i) and (ii) above. By condition

(iii) above

$$\tilde{I}_1(\bar{M}_j - (p_1+t)_j^{\min}) = a + b(\bar{M}_j - (p_1+t)_j^{\min}) \text{ where } b > 0, \\ \text{for all } j \geq 1 \quad 2-7-17$$

$$= a + b(M - TE_j(\delta^*)) \text{ for all } j \geq 1 \text{ by}$$

$$(2-7-16) \quad 2-7-18$$

Substituting (2-7-18) into (2-7-15) gives

$$V_0(\xi^*, \delta^*) = \int_{p_1^L}^{p_1^U} \max[a + b(M - TE_1(\delta^*))], \int_{p_1^L}^{p_1^U} \max[a + b(M - TE_2(\delta^*)), \dots, \\ \int_{p_1^L}^{p_1^U} \max[a + b(M - TE_j(\delta^*))], \int_{p_1^L}^{p_1^U} \max[a + b(M - TE_{j+1}(\delta^*)), \dots] f_g(p_1^{j+1} | y_j) dp_1^{j+1}] \\ f_g(p_1^j | y_{j-1}) dp_1^j] \dots] f_g(p_1^2 | y_1) dp_1^2] f_g(p_1^1) dp_1^1 \quad 2-7-19$$

$$\begin{aligned}
&= a + bM - b \int_{p_1^L}^{p_1^U} \min[TE_1(\delta^*), \int_{p_1^L}^{p_1^U} \min[TE_2(\delta^*), \dots, \int_{p_1^L}^{p_1^U} \min[TE_j(\delta^*), \\
&\int_{p_1^L}^{p_1^U} \min[TE_{j+1}(\delta^*), \dots] f_g(p_1^{j+1} | y_j) dp_1^{j+1}] f_g(p_1^j | y_{j-1}) dp_1^j] \dots] f_g(p_1^2 | y_1) dp_1^2] \\
&\quad \cdot f_g(p_1^1) dp_1^1 \quad 2-7-20
\end{aligned}$$

$$= a + bM - bW_0(\xi^*, \delta^*) \text{ by (2-6-6)} \quad 2-7-21$$

$(a + bM)$ is a constant and Proposition 2-5-5 proved $V_0(\xi^*, \delta^*)$ to be a maximum with respect to the choice of sequential search rule $\rho_s = (\xi, \delta)$. Hence, by (2-7-21), $W_0(\xi^*, \delta^*)$ is a minimum with respect to the choice of sequential search rule $\rho_s = (\xi, \delta)$, ie.

$$W_0(\xi^*, \delta^*) \leq W_0(\xi, \delta) \text{ for any sequential search rule } \rho_s = (\xi, \delta). \quad 2-7-22$$

However, by definition (2-2-22),

$$W_0(\hat{\xi}, \hat{\delta}) \leq W_0(\xi, \delta) \text{ for any sequential search rule } \rho_s = (\xi, \delta). \quad 2-7-23$$

Therefore, from (2-7-22) and (2-7-23),

$$W_0(\xi^*, \delta^*) = W_0(\hat{\xi}, \hat{\delta}) \quad 2-7-24$$

Conditions (i), (ii) and (iii) are therefore sufficient for the validity of the expenditure minimisation proxy.

(b) Necessity: $\rho_s^* = (\xi^*, \delta^*)$ possesses the property that

$$V_0(\xi^*, \delta^*) \geq V_0(\xi, \delta) \text{ and } W_0(\xi^*, \delta^*) \leq W_0(\xi, \delta) \quad 2-7-25$$

for any sequential search rule $\rho_s = (\xi, \delta)$. For (2-7-25) to be true

$W_0(\xi^*, \delta^*)$ must exist. The existence of $W_0(\xi^*, \delta^*)$ is conditional upon a unit of commodity 1 always being purchased. Conditions necessary for (2-7-25) are therefore that the consumer always enters the market for commodity 1, that he always demands one unit of commodity 1 and that this demand is always feasible, ie.

$$(i) \quad U \in \Omega \quad 2-7-26$$

$$(ii) \quad x_1^* \equiv 1 \text{ is always feasible.} \quad 2-7-27$$

The expected total expenditure minimising consumer is risk neutral with respect to wealth net of total expenditure on commodity 1. This requires

- (iii) \tilde{I}_1 is linear with respect to $(M - TE_j(\delta^*)) = \bar{M}_j - (p_1 + t)_j^{\min}$ 2-7-28
for all $j \geq 1$. Q.E.D.

The conditions required by Proposition 2-7-1 are clearly very stringent and unusual. Consequently minimisation of ex ante expected total expenditure on the purchase of a unit of commodity 1 will usually not be a valid proxy for maximisation of ex ante expected utility. One must disagree with Rothschild [43, p.690] who, as a justification for ignoring psychic costs of search, says "Other rather trivial, from the formal point of view, generalisations are possible. Price and cost can be measured in utility rather than money. However, the utility function must be linear, so this is a small generalisation". This paper demonstrates that measuring price and cost in utility terms is *not* a "small generalisation" [43, p.690]. Instead it is a generalisation necessary to avoid the contradictions inherent in the stopping rule (2-1-1) except under the stringent conditions of Proposition 2-7-1. Nevertheless, minimisation of expenditure on commodity 1 is the consumer objective invoked in Chapters 3 and 4 because using a more general utility maximising objective would greatly complicate the analyses contained in these chapters.

Note that transaction costs are included in the expenditure minimising search rule $\hat{\rho}_s$ described in Section 2-6 and that this rule need not be myopic. For $\hat{\rho}_s$ to be equivalent to the Rothschild, Telser, Axell, Gastwirth search rules the already restrictive conditions of Proposition 2-7-1 must be augmented by a condition of zero transaction costs and conditions sufficient for $\hat{\rho}_s$ to be myopic.

One apparent contradiction must be resolved. It is asserted that the consumer's indirect utility function I must be linear w.r.t. $(\bar{M}_j - (p_1 + t)_j^{\min})$ for all $j \geq 1$ for the optimal sequential search rule ρ_S^* to coincide with the sequential search rule minimising expected total expenditure on commodity 1. The consumer is therefore risk neutral w.r.t. total expenditure on commodity 1. Yet, the stopping rule ξ^* seems to imply some degree of risk averseness in that

$\xi_j^*(y_j) = S_j^* = 1$ if $TE_j(\delta^*) = E[W_{j+1}(\xi^*, \delta^*) | y_j]$. Therefore, if

$$TE_j(\delta^*) = E[TE_{j+k}(\delta^*) | y_j] = E[W_{j+1}(\xi^*, \delta^*) | y_j] \text{ for some } k \geq 1 \quad 2-7-29$$

the searcher will always decline the fair gamble of continuing his search past his present position in favour of stopping his search now.

The "contradiction" does not in fact exist, because, as Proposition 2-7-2 proves, the consumer will be indifferent between stopping and continuing his search only with probability zero. Only two-way ties of the type (2-7-29) are considered in the proof of Proposition 2-7-2 because the probability of any higher order tie is bounded above by the probability of a two-way tie. Proving the probability of a two-way tie is zero therefore automatically proves that the probability of any higher order tie is also zero. Proposition 2-7-2's result is proved under conditions which ensure that expenditure minimisation is a valid proxy for utility maximisation. The result will not be generally true with a more general objective of maximising ex ante expected utility since the frequency with which indifference occurs through the use of the optimal sequential search rule will depend upon the form of the consumer's utility function.

Proposition 2-7-2:

If the conditions of Proposition 2-7-1 are satisfied, the searcher will be indifferent between stopping or continuing his search

at any stage only with probability zero.

Proof:

Suppose the consumer has taken j observations on p_1 and finds he is indifferent between stopping and taking an additional observation on p_1 because a tie such as (2-7-29) has occurred, ie.

$$E[W_{j+1}(\xi^*, \delta^*) | y_j] = TE_j(\delta^*) = E[TE_{j+k}(\delta^*) | y_j] \quad 2-7-30$$

for some $k \geq 1$. The probability of this tie occurring will be shown to be zero. From (2-7-30),

$$(p_1 + t)_j^{\min} + c(j) = E[(p_1 + t)_{j+k}^{\min} + c(j+k) | y_j] \quad 2-7-31$$

$$(p_1 + t)_j^{\min} - E[(p_1 + t)_{j+k}^{\min} | y_j] = \sum_{i=j+1}^{j+k} c_i \quad 2-7-32$$

$$\text{Now, } \Pr((p_1 + t)_{j+k}^{\min} \geq \bar{p}_1 | y_j) = \Pr\left(\bigcap_{i=1}^k \{p_1^{j+i} + t_{j+i} \geq \bar{p}_1\} | y_j\right) \quad 2-7-33$$

$$= \Pr\left(\bigcap_{i=1}^k \{p_1^{j+i} \geq \bar{p}_1 - t_{j+i}\} | y_j\right) \quad 2-7-34$$

The p_1^{j+i} , $i=1, \dots, k$, are independently distributed (assumption (1-4-10)) so

$$\Pr((p_1 + t)_{j+k}^{\min} \geq \bar{p}_1 | y_j) = \prod_{i=1}^k (1 - F_g(\bar{p}_1 - t_{j+i})) \quad 2-7-35$$

$$\text{Now, } \Pr((p_1 + t)_{j+k}^{\min} = \bar{p}_1 | y_j) = d/d\bar{p}_1 (\Pr((p_1 + t)_{j+k}^{\min} \leq \bar{p}_1 | y_j)) \quad 2-7-36$$

so from (2-7-36) and (2-7-35),

$$\Pr((p_1 + t)_{j+k}^{\min} = \bar{p}_1 | y_j) = \sum_{i=1}^k \prod_{\substack{\ell=1 \\ \ell \neq i}}^k (1 - F_g(\bar{p}_1 - t_{j+\ell})) \cdot f_g(\bar{p}_1 - t_{j+i}) \quad 2-7-37$$

Substituting (2-7-37) into (2-7-32) gives

$$\begin{aligned} (p_1 + t)_j^{\min} - (p_1 + t)_j^{\min} & \int_{(p_1 + t)_j^{\min}}^{\bar{p}_1} \sum_{i=1}^k \prod_{\substack{\ell=1 \\ \ell \neq i}}^k (1 - F_g(\bar{p}_1 - t_{j+\ell})) \cdot f_g(\bar{p}_1 - t_{j+i}) d\bar{p}_1 \\ & - \int_{\bar{p}_1}^L (p_1 + t)_j^{\min} \bar{p}_1 \cdot \sum_{i=1}^k \prod_{\substack{\ell=1 \\ \ell \neq i}}^k (1 - F_g(\bar{p}_1 - t_{j+\ell})) \cdot f_g(\bar{p}_1 - t_{j+i}) d\bar{p}_1 = \sum_{i=j+1}^{j+k} c_i \quad 2-7-38 \end{aligned}$$

Rearranging (2-7-38) gives

$$\int_{p_1}^{(p_1+t)_j^{\min}} ((p_1+t)_j^{\min} - \bar{p}_1) \cdot \sum_{i=1}^k \prod_{\substack{\ell=1 \\ \ell \neq i}}^k (1 - F_g(\bar{p}_1 - t_{j+\ell})) \cdot f_g(\bar{p}_1 - t_{j+i}) d\bar{p}_1 \\ = \sum_{i=j+1}^{j+k} c_i \quad 2-7-39$$

The L.H.S. of (2-7-39) is a monotonic increasing function in $(p_1+t)_j^{\min}$ so there is only one solution, $(p_1+t)_j^{\min} = p^*$, to (2-7-39). However, the p.d.f. of $(p_1+t)_j^{\min}$ is continuous so the probability that $(p_1+t)_j^{\min} = p^*$ exactly is zero. Hence the probability of the tie (2-7-30) is zero.

Q.E.D.

In Section 2-1 it was mentioned that the search rule (2-1-1) possesses a 'supermartingale' structure which makes the search rule 'myopic' in that the stopping rule component only considers the expected expenditure on commodity 1 after one extra observation on p_1 . In the more general structure of ρ_s^* developed in Sections 2-3 to 2-5, ξ^* considers the expected utilities after one, two, ..., etc. extra observations on p_1 , assesses which of these expectations is the greatest and continues or halts search if this maximum does or does not exceed the utility presently attainable. A supermartingale structure is therefore one in which, after a certain number of observations on p_1 , the utility expected after one additional observation on p_1 exceeds the utility expected after two or more observations on p_1 . The reader is referred to De Groot [9, p.353] or McCall [30, p.423] for formal discussions on the properties of martingales.

Definition:

The optimal sequential search rule $\rho_s^* = (\xi^*, \delta^*)$ is myopic if and only if

$$E[V_{j+1}(\xi^*, \delta^*) | y_j] = E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} | y_j] \quad 2-7-40$$

for all $j \geq j^*$ where $j^* \geq 0$.

Whether or not ρ_S^* possesses the myopic structure (2-7-40) largely depends on the nature of the costs facing the searcher, a dependency examined in the discussion on search paths in Chapter 3.

Under the supermartingale structure (2-7-40), the j th component of the optimal sequential stopping rule greatly simplifies from (2-3-9) to

$$\xi_j^*(y_j) = S_j^* = \begin{cases} 0, & \max\{U_j^*(0), U_j^*(1)\} < E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} | y_j] \\ 1, & \max\{U_j^*(0), U_j^*(1)\} \geq E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} | y_j] \end{cases} \quad 2-7-41$$

Since $U_{j+1}^*(0) < U_j^*(0)$ for all $j \geq 0$, (2-7-41) simplifies further to

$$\xi_j^*(y_j) = S_j^* = \begin{cases} 0, & \max\{U_j^*(0), U_j^*(1)\} < E[U_{j+1}^*(1) | y_j] \\ 1, & \max\{U_j^*(0), U_j^*(1)\} \geq E[U_{j+1}^*(1) | y_j] \end{cases} \quad 2-7-42$$

The myopia contained in the stopping rule is apparent from (2-7-42)

since the decision of whether or not to take an extra observation on p_1 is made by comparing the presently attainable utility with the utility expected to be attainable after only one extra observation on p_1 . If the conditions of Proposition 2-7-1 are satisfied as well as (2-7-40),

$$E[W_{j+1}(\xi^*, \delta^*) | y_j] = E[TE_{j+1}(\delta^*) | y_j] \quad 2-7-43$$

and the optimal stopping rule ξ^* has the particularly simple structure of

$\xi^* = (\xi_0^*, \xi_1^*, \dots, \xi_j^*, \dots)$ where $\xi_0^*(y_0) = S_0^* \equiv 0$ and

$$\xi_j^*(y_j) = S_j^* = \begin{cases} 0, & TE_j(\delta^*) > E[TE_{j+1}(\delta^*) | y_j] \\ 1, & TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \end{cases} \quad \text{for all } j \geq 1 \quad 2-7-44$$

If marginal financial search costs are a constant c and all transaction costs are zero, (2-7-44) is precisely the simple rule (2-1-1).

Myopia will be optimal if and only if the conditions of search are such that (2-7-40) is true. The conditions required for the simple search rule (2-1-1) to coincide with the optimal sequential search rule ρ_s^* are one set of conditions sufficient for myopic search to be optimal. However, more general sufficient conditions than these exist. In Section 3-7 it is shown that the rational consumer chooses to contact sellers in a sequence (a search path) for which (2-7-40) is satisfied. Myopic search behaviour thus coincides with optimal search behaviour more frequently than it might at first appear.

The search model developed in Sections 2-2 to 2-5 contains assumption (2-2-18), that a search path is decided upon prior to search and is not revised as search proceeds. There are real life situations where a consumer is constrained to exactly this course of action. However, if he expects to attain greater utility by changing his search path, and the freedom exists for him to do so, then he will change his search path. Chapter 3 examines how a search path is chosen, some of the factors influencing this choice and how changes in these factors can change the chosen path. A description is given of how a sequential searcher's choice of search path may be revised as search proceeds.

CHAPTER III

THE DIRECTION OF SEARCH AND OPTIMAL SEARCH PATHS

SECTION 3-1: WHY SEARCH PATHS MATTER

In real life, searchers do not mill aimlessly about among sellers. Rational search is always directed in some way. The actual direction will depend upon the searcher's knowledge of different sellers' existences and characteristics, the relative costs and benefits from contacting different sellers, the constraints imposed on the searcher by the search problem he faces and the manner in which he learns about all of these. It may well be that these constraints are such that only particular sellers can be contacted in one particular sequence but casual observation of reality suggests that the constraints upon a searcher's search path are rarely this severe. Usually, therefore, a searcher will have some freedom to choose the sellers he contacts and to choose the order in which he contacts them. To ignore the variations in the expected net gain from search due to variations in these sequences makes as little sense as ignoring the variations in a firm's expected profit due to varying the mix of factor inputs. Just as the competitive firm optimises by choosing a mix of factor inputs which maximise profits subject to the constraints on the firm, the rational searcher will choose a search path which maximises the expected gain from search, subject to the constraints placed upon him by his search problem. Yet, with two exceptions, the search literature is devoid of any consideration of search paths. The exceptions are papers by Burdet [2] and Salop [46]. This writer has been unable to obtain a copy of Burdet's

paper and so cannot describe his results. Salop considers a job searcher who must accept a job within a finite time horizon in a labour market with dispersed wages. The searcher has subjective uncertain prior knowledge of the likelihood of being offered a job by any particular firm and of the wage rate accompanying such an offer. Salop assumes that this stock of information is not altered over the course of the search. One firm is sampled per period. If the sequential searcher refuses a firm's offer, an opportunity cost is incurred of the wage offered by the firm for one period. Salop shows the job searcher, who has an objective of maximising his present valued expected wealth, finds it optimal to sample firms in a specific order. The order of sampling is identical to the ranking of firms in a decreasing sequence of the wealth the searcher expects to receive from the individual firms. The searcher's acceptance wage is shown to decline as search proceeds.

The implicit assumption typically present in the search literature is that, for some reason never specified, the searcher is indifferent between all possible search paths or, alternatively, that only one search path is possible. While the simplicity achieved in the analysis of search models by assuming indifference between possible search paths is useful, it is achieved only at the expense of being unable to glean any insights into how a search path is chosen. The purpose of this chapter is to determine some of the factors influencing this choice.

Only sequential search paths are considered in this chapter. Stigleresque search paths are considered in Section 5-4.

The following sections examine the influences of financial search costs, transaction costs, searcher's knowledge of sellers' relative pricing behaviour and advertising on the choice of search

path. The first three factors are shown to give rise to supermartingale sequences of net expected gains from search, so that myopic search is optimal (see Section 2-7), when marginal financial search costs are independent.

SECTION 3-2: NOTATION AND ASSUMPTIONS

Throughout the whole of the chapter it is assumed that the conditions of Proposition 2-7-1 are satisfied so that minimisation of expenditure on commodity 1 is a valid proxy for consumer utility maximisation. It is recognised that these conditions are restrictive, but the simplification gained by the use of the proxy is considerable and allows at least an intuitive understanding of what may motivate the searcher in his choice of search path when he has a more general objective, such as maximising expected utility.

Assumption; Minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility.

3-2-1

The analysis in this chapter is therefore carried out in money terms.

A unique index is attached to each seller so that one seller may be identified from any other. It is supposed that each index is a positive integer. No two sellers may have the same index. The index does not imply a ranking of any sort over the sellers - it is simply a means of identifying particular sellers. For instance, the clothing store two blocks away may have an index of 14076 while the clothing store four kilometres due south may have an index of 32. The indices are used to indicate which sellers the searcher is considering contacting.

i_1, \dots, i_j - a search path over j sellers, $j \geq 1$, is defined by a

sequence of j indices, i_1, \dots, i_j , where $i_k = n$ means that the 3-2-2
 k th seller a searcher would contact by following this search path is
the seller attached to an index of n . For example, a search path over
 $j = 2$ sellers may be 14076, 32, meaning the clothing store two blocks
away would be contacted first and the store four kilometres due south
contacted second. A potential point of misunderstanding must be
emphasised here. Choosing a search path i_1, \dots, i_j does not, in
general, commit a searcher to contacting all of the sellers associated
with these j indices in the order i_1, \dots, i_j . Assumption (3-2-1)
guarantees that a consumer always searches. With a search path of
 i_1, \dots, i_j the first seller he contacts is the seller attached to the
index i_1 . If the searcher decides to take a second observation on p_1
there is, in general, no guarantee that the second seller contacted is
the seller attached to the index i_2 . The selling price quoted by the
first seller, p_1^1 , may alter the consumer's choice of search path to
 i'_1, \dots, i'_j , where j' need not equal j and i'_k need not equal i_k for
any of $k = 2, \dots, j$. The choice of a particular search path at a
particular decision point thus fixes definitely only the identity of the
very next seller to be contacted should search be extended by an additional
observation on p_1 . The actual search path followed is known with
certainty to the searcher only at the moment he stops searching. However,
Section 3-3 shows that if marginal costs of observations are independent,
then, while the searcher may choose to vary the length of his search
path, he will never choose to vary the order in which sellers on this
search path are contacted. The searcher's preferences over all sellers
on the order of contacting sellers will not change as search proceeds.

Thus if second, third and subsequent observations are taken on p_1 , the sellers from whom these observations are taken are still the sellers with indexes of i_2 , i_3 etc. on the original search path.

In Section 2-2, assumption (2-2-18) prevented consideration of alternative search paths. Lifting the assumption in this chapter raises a difficult problem which may be avoided by the following assumption.

Assumption: The marginal financial cost of any observation on p_1 is independent of any characteristic of any other observation on p_1 .

3-2-3

(3-2-3) will be recognised as part of assumption (2-1-3), implicit in the search models discussed in Section 2-1. The difficult problem being avoided may be inferred from the following example. Suppose a rural housewife wishes to buy a dress. There is a local drapery three miles down the road while ten miles in the other direction is a city with many draperies. As far as the housewife knows there is no difference in the relative pricing behaviour or transaction costs of any of these sellers. If she decides it is likely she will wish to accept the first selling price offered to her, it is intuitively reasonable for her to visit the local draper first, since the cost of this observation is less than the cost of obtaining a first observation from a city draper. However, if she feels it is likely that she will obtain more than one quotation, it may be more profitable to avoid the local draper and go straight to the city drapers. While the marginal cost of the first observation at a city draper is greater than for the local draper, the marginal costs of subsequent observations are lower since city drapers

are clustered closely together. The difficulty is that the marginal cost of an observation is dependent upon the location of at least the immediately preceding observation. Computation of the optimal searcher strategy for such problems requires the solution of a very large, very complex dynamic programming problem. The complexity of this problem is substantially reduced and great analytical simplification obtained, for only a little loss in understanding of characteristics of optimal searcher strategies, if assumption (3-2-3) is invoked.

Recall from Section 1-4 that c_{i_j} is the marginal financial search cost of obtaining a selling price quotation from the j th seller on the searcher's chosen search path and that t_{i_j} is the transaction cost of purchasing a unit of commodity 1 from the j th seller on the searcher's chosen search path.

The expected total expenditure on commodity 1 associated with a particular search path i_1, \dots, i_j depends upon the relative likelihoods of search being halted after any of 1 to j observations on p_1 have been taken.

$\Pr(\psi_k^*=1 | w_{i_1}, \dots, w_{i_j}, t_{i_1}, \dots, t_{i_j}, c_{i_1}, \dots, c_{i_j})$ - the ex ante probability 3-2-4 of the searcher wishing to halt search after k , $1 \leq k \leq j$, observations on p_1 given a chosen search path i_1, \dots, i_j . To ensure the probabilities $\Pr(\psi_1^*=1 | w_{i_1}, \dots, c_{i_j})$ to $\Pr(\psi_j^*=1 | w_{i_1}, \dots, c_{i_j})$ form a complete p.m.f. of search lengths for the chosen search path, $\Pr(\psi_j^*=1 | w_{i_1}, \dots, c_{i_j})$ is defined as

$$\Pr(\psi_j^*=1 | w_{i_1}, \dots, c_{i_j}) = 1 - \sum_{k=1}^{j-1} \Pr(\psi_k^*=1 | w_{i_1}, \dots, c_{i_j}) \quad 3-2-5$$

ie. the searcher assesses the ex ante expected total expenditure on commodity 1 for the search path i_1, \dots, i_j on the basis that the

searcher may not proceed past seller i_j , if he has not already stopped his search before reaching this seller.

These probabilities depend upon the values of the observations on p_1 and are therefore revised as search proceeds and observations on p_1 accumulate. Revision of these probabilities may well result in the searcher changing his choice of search path as search proceeds.

$ETE(i_1, \dots, i_j)$ - the ex ante expected total expenditure on commodity 1

for the search path i_1, \dots, i_j . 3-2-6

$$ETE(i_1, \dots, i_j) = E \left[\sum_{k=1}^j \{(p_1 + t)_k^{\min} + \sum_{m=1}^k c_{i_m}\} \cdot \psi_k^* | w_{i_1}, \dots, w_{i_j} \right] \quad 3-2-7$$

The searcher evaluates $ETE(i_1, \dots, i_j)$ for all possible search paths i_1, \dots, i_j and, since minimisation of expenditure on commodity 1 is equivalent to utility maximisation, chooses the search path for which ETE is a minimum.

$i_1^*, \dots, i_{j^*}^*$ - the searcher's optimal search path ie.

$$ETE(i_1^*, \dots, i_{j^*}^*) \leq ETE(i_1, \dots, i_j) \quad 3-2-8$$

for any search path i_1, \dots, i_j .

The reader is reminded that the ex ante optimal search path $i_1^*, \dots, i_{j^*}^*$ neither commits the searcher to a search of j^* observations on p_1 nor commits him to contacting any of the sellers with the indices $i_2^*, \dots, i_{j^*}^*$. Since at least one observation on p_1 is always taken, choosing the search path $i_1^*, \dots, i_{j^*}^*$ does commit the searcher to contacting the seller attached to the index i_1^* , but this is the only firm commitment implied by this choice of search path.

The results obtained in Sections 3-3, 3-4 and 3-5 utilise rankings over all the possible values of the c_{i_j} , t_{i_j} and w_{i_j} . The

following four definitions are given in preparation for these results.

- $c_{(j)}$ - the j th smallest member of the set of all marginal financial search costs $c_{i,j}$ ie. $c_{(1)} \leq c_{(2)} \leq \dots \leq c_{(j)} \leq \dots$ for all $c_{i,j}$. 3-2-9
- $t_{(j)}$ - the j th smallest member of the set of all transaction costs $t_{i,j}$ ie. $t_{(1)} \leq t_{(2)} \leq \dots \leq t_{(j)} \leq \dots$ for all $t_{i,j}$. 3-2-10
- $w_{(j)}$ - the j th smallest member of the set of all values of $w_{i,j}$ of w ascribed to sellers ie. $w_{(1)} \leq w_{(2)} \leq \dots \leq w_{(j)} \leq \dots$ for all $w_{i,j}$. 3-2-11
- $\bar{w}_{(j)}$ - the j th largest member of the set of all values of $w_{i,j}$ of w ascribed to sellers ie. $\bar{w}_{(1)} \geq \bar{w}_{(2)} \geq \dots \geq \bar{w}_{(j)} \geq \dots$ for all $w_{i,j}$. 3-2-12

In Sections 3-3, 3-4 and 3-5 respectively, the directional influences of financial search costs, transaction costs and searcher's knowledge of sellers' relative pricing behaviour are examined. Section 3-6 combines these influences in a discussion on advertising. In Section 3-7 the results of Sections 3-3, 3-4 and 3-5 are used to show that following an optimal search path will cause myopic and optimal sequential search behaviours to be equivalent.

SECTION 3-3: FINANCIAL SEARCH COSTS AND THE OPTIMAL SEARCH PATH

It is intuitive that financial search costs influence both the direction and duration of consumer search. To assist in understanding how financial search costs influence the choice of search path, directional influences due to transaction costs and searcher's beliefs about sellers' relative pricing behaviours are assumed not to exist.

Assumption: There are no transaction costs ie. $t_{i_j} = 0$ for all i_j . 3-3-1

Assumption: The searcher has no knowledge of differences in different sellers' relative pricing behaviour ie. $w_{i_j} = w$ for all i_j . 3-3-2

Under assumptions (3-3-1) and (3-3-2), the p.m.f. of search lengths (3-2-4) and the ex ante expected total expenditure on commodity 1 (3-2-7) for a search path i_1, \dots, i_j simplify to

$$\Pr(\psi_k^*=1 | w_{i_1}, \dots, w_{i_j}, c_{i_1}, \dots, c_{i_j}) = \Pr(\psi_k^*=1 | w, c_{i_1}, \dots, c_{i_j}) \quad 3-3-3$$

for all $k = 1, \dots, j$, and

$$\text{ETE}(i_1, \dots, i_j) = E \left[\sum_{k=1}^j \{p_{1k}^{\min} + \sum_{m=1}^k c_{i_m}\} \cdot \psi_k^* | w \right] \quad 3-3-4$$

In a sequential search context the terms "length of search" and "number of observations on p_1 " are interchangeable, although this is not the case in the context of generalised search where many observations on p_1 may be made in any one period. The term commonly used in the search literature is "length of search" so, to facilitate comparison of other writers' analyses to the present work, the term will be retained throughout Chapters 3 and 4 where sequential search is optimal. Nevertheless, writers should distinguish properly between the periodic duration of search and the total number of observations made.

The analysis of this section begins by arbitrarily selecting a particular length of search j . The conditionally optimal search path of length j is found to be characterised by a monotonic increasing sequence of the smallest j marginal financial search costs. The optimal search path is then obtained by locating the search length j^* corresponding to the global minimum for all values of $j \geq 1$ of the minima for each of the values of j . The search length for which the ex ante expected expenditure on commodity 1 is a global minimum need not

be the actual length of search. The problem analysed here, therefore, is one in which the searcher may vary the number of observations on p_1 , the physical method of search, the sellers contacted and the order in which they are contacted. The actual length of search is known only at the instant the searcher halts his search.

It is intuitive that a searcher will wish to incur only the smallest marginal financial search costs. It is also intuitive that the relatively larger is a marginal cost, the later in his search is the point at which the searcher will wish to incur it. This is because there will then be a greater chance that the searcher will have found it optimal to halt search before incurring the cost and so avoid it entirely. This intuition is formalised in the following Proposition.

Proposition 3-3-1:

A search path of length $j \geq 1$ for which the ex ante expected total expenditure on commodity 1 is minimised is i_1^*, \dots, i_j^* where $c_{i_m^*} = c_{(m)}$ for all $m = 1, \dots, j$.

Proof:

The ex ante expected total expenditure on commodity 1 for a search path i_1, \dots, i_j is, by (3-3-4),

$$ETE(i_1, \dots, i_j) = \sum_{k=1}^j \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_k^* \cdot \{p_{1k}^{\min} + \sum_{m=1}^k c_{i_m}\} \cdot f_g(p_1^1, \dots, p_1^j) dp_1^1 \dots dp_1^j$$

3-3-5

By analogy to (2-6-6) and (2-5-13), (3-3-5) may be written as

$$ETE(i_1, \dots, i_j) = \int_{p_1^L}^{p_1^U} \min\{p_{11}^{\min} + c_{i_1}, \int_{p_1^L}^{p_1^U} \min\{p_{12}^{\min} + c_{i_1} + c_{i_2}, \int_{p_1^L}^{p_1^U} \min\{p_{13}^{\min} + c_{i_1} + c_{i_2} + c_{i_3}, \dots\} f_g(p_1^3 | y_2) dp_1^3\} f_g(p_1^2 | y_1) dp_1^2\} f_g(p_1^1) dp_1^1$$

3-3-6

The problem is to select the values of the c_{i_m} for which (3-3-6) is minimised. Since c_{i_1} is a component of the total expenditures at all stages of search 1 to j ,

$$\text{ETE}(i_1, \dots, i_j) \geq \int_{p_1^L}^{p_1^U} \min\{p_{11}^{\min} + c_{(1)}, \int_{p_1^L}^{p_1^U} \min\{p_{12}^{\min} + c_{(1)} + c_{i_2},$$

$$\int_{p_1^L}^{p_1^U} \min\{p_{13}^{\min} + c_{(1)} + c_{i_2} + c_{i_3}, \dots\} f_g(p_1^3 | y_2) dp_1^3 \} f_g(p_1^2 | y_1) dp_1^2 \} f_g(p_1^1) dp_1^1$$

3-3-7

$$\geq \int_{p_1^L}^{p_1^U} \min\{p_{11}^{\min} + c_{(1)}, \int_{p_1^L}^{p_1^U} \min\{p_{12}^{\min} + c_{(1)} + c_{(2)}, \int_{p_1^L}^{p_1^U} \min\{p_{13}^{\min}$$

$$+ c_{(1)} + c_{(2)} + c_{i_3}, \dots\} f_g(p_1^3 | y_2) dp_1^3 \} f_g(p_1^2 | y_1) dp_1^2 \} f_g(p_1^1) dp_1^1$$

3-3-8

$\geq \dots\dots\dots$

$$\geq \text{ETE}(i_1^*, \dots, i_j^*)$$

3-3-9

for any search path i_1, \dots, i_j .

Q.E.D.

The characterisation of the optimal search path $i_1^*, \dots, i_{j^*}^*$ in terms of marginal financial search costs follows immediately from Proposition 3-3-1.

Proposition 3-3-2:

An optimal search path $i_1^*, \dots, i_{j^*}^*$ is such that

(i) $c_{i_m^*} = c_{(m)}$ for all $m = 1, \dots, j^*$

(ii) $ETE(i_1^*, \dots, i_{j^*}^*) \leq ETE(i_1, \dots, i_j)$

for any search path i_1, \dots, i_j .

Proof:

(ii) above follows from definition (3-2-8). An optimal search path minimises the ex ante expected expenditure on commodity 1.

Hence Proposition 3-3-1 may be applied to establish (i) above.

Q.E.D.

The ex ante optimal search path will be unique if and only if no two of the marginal financial costs incurred on the path are equal. If any two are equal then the seller will be indifferent as to the order in which he solicits quotations from the sellers for whom the marginal costs are the same. The searcher will therefore be indifferent between the two search paths containing these orderings. If all marginal financial search costs are equal, $c_{i_k} = c$ for all i_k , then the searcher will be indifferent to any search path of the ex ante optimal search length.

Section 2-1 comments that the physical manner in which search is conducted is an important factor determining the magnitude of the marginal financial costs of search. Faced with a variety of search methods, the rational searcher will selectively utilise the method or methods for which the marginal financial costs are least. For instance, if only two sellers have telephones and the marginal financial costs of search are least if the search method employed is telephoning, then these two sellers will be the first two sellers on the searcher's optimal search path. If the searcher desires more than two quotations

he must alter his method of search, perhaps to visiting sellers in person, for his third and subsequent observations. The marginal financial costs of these observations are consequently greater than the first two marginal costs.

Increases in marginal financial search costs are increases in disincentives for search. One expects the number of observations taken on p_1 to decline as marginal financial search costs increase. This is a reason for sellers to ensure they are accessible to consumers. A seller presenting a smaller marginal search cost to a searcher will appear in an earlier position on the searcher's search path and so is more likely to have the searcher ask him for a quotation.

Kohn and Shavell [20, p. 112, theorem 14] prove that the reservation price of a searcher at any stage of his search declines as the marginal cost of the next observation rises, and that the reservation price cannot rise with an increase in the marginal cost of any future observation. They comment that lower reservation prices mean a lowering in the expected length of search. In Section 4-3 the exact form of the dependence of the ex ante expected search length on marginal financial search costs is derived. In this section, the exact result is foreshadowed by Proposition 3-3-3 which shows that increasing the k th marginal financial search cost, $2 \leq k \leq j^*$, may increase (cannot decrease) the ex ante probability of the searcher wishing to halt his search before taking k observations on p_1 and may decrease (cannot increase) the ex ante probability of the searcher wishing to halt his search after taking k or more observations on p_1 . Consequently the ex ante expected length of search cannot be increased by an increase in any of the marginal financial search costs.

It is shown in Proposition 3-3-3 that the ex ante probabilities of search length are independent of the first marginal financial search cost $c_{i_1^*}$, although the ex ante expected expenditure on commodity 1 is always dependent upon $c_{i_1^*}$. The independence is caused by two factors, both of which arise from imposing assumption (3-2-1) that the conditions of Proposition 2-7-1 are satisfied. Assumption (3-2-1) guarantees a searcher always searches and that his objective is to minimise his expenditure on commodity 1. Always searching means $c_{i_1^*}$ is always incurred, no matter how many observations the searcher makes on p_1 . $c_{i_1^*}$ is, therefore, always a component of any total expenditure on commodity 1 made by the searcher and thus has no effect in determining when one such expenditure is less than all other such expenditures. Consequently $c_{i_1^*}$ does not influence any of the probabilities of search length.

Proposition 3-3-3:

$$\frac{\partial \Pr(\psi_j^*=1 | w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_k^*}} \begin{cases} \geq 0; & \text{if } k \geq j+1, j=1, \dots, j^*-1 \\ = 0; & \text{if } k = 1, j = 1, \dots, j^* \\ \leq 0; & \text{if } k \leq j, j = 2, \dots, j^* \end{cases}$$

Proof:

The ex ante probability of a search length of j , $1 \leq j \leq j^*$, is

$$\Pr(\psi_j^*=1 | w, c_{i_1^*}, \dots, c_{i_{j^*}^*}) = \Pr(TE_j < TE_n, \text{ for all } n \neq j, 1 \leq n \leq j^* | w, c_{i_1^*}, \dots, c_{i_{j^*}^*}) \quad 3-3-10$$

$$= \Pr(p_{1j}^{\min} + \sum_{m=1}^j c_{i_m^*} < p_{1n}^{\min} + \sum_{m=1}^n c_{i_m^*}, \text{ for all } n \neq j, 1 \leq n \leq j^* | w) \quad 3-3-11$$

$$= \Pr\left(\bigcap_{\substack{n=1 \\ n \neq j}}^{j^*} \{p_{1j}^{\min} + \sum_{m=1}^j c_{i_m^*} < p_{1n}^{\min} + \sum_{m=1}^n c_{i_m^*}\} \mid w\right) \quad 3-3-12$$

$$p_{1j}^{\min} = \min(p_1^1, \dots, p_1^j) \text{ so } p_{1j}^{\min} \geq p_{1n}^{\min} \text{ if and only if } j \leq n \quad 3-3-13$$

$$\text{and } p_{1j}^{\min} \leq p_{1n}^{\min} \text{ if and only if } j \geq n \quad 3-3-14$$

Inserting (3-3-13) and (3-3-14) into (3-3-12) gives

$$\begin{aligned} & \Pr(\psi_j^*=1 \mid w, c_{i_1^*}, \dots, c_{i_{j^*}^*}) \\ &= \Pr\left(\left[\bigcap_{n=1}^{j-1} \{p_{1n}^{\min} - p_{1j}^{\min} > \sum_{m=n+1}^j c_{i_m^*}\}\right] \cap \left[\bigcap_{n=j+1}^{j^*} \{p_{1j}^{\min} - p_{1n}^{\min} < \sum_{m=j+1}^n c_{i_m^*}\}\right] \mid w\right) \quad 3-3-15 \end{aligned}$$

Note that $c_{i_1^*}$ does not appear in (3-3-15) for the two reasons given before this Proposition. Since $c_{i_1^*}$ does not influence the p.m.f. of search lengths,

$$\frac{\partial \Pr(\psi_j^*=1 \mid w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_1^*}} = 0 \text{ for all } j = 1, \dots, j^* \quad 3-3-16$$

If $k \geq j+1$, an increase in $c_{i_k^*}$ influences the event

$$\bigcap_{n=j+1}^{j^*} \{p_{1j}^{\min} - p_{1n}^{\min} < \sum_{m=j+1}^n c_{i_m^*}\} \quad 3-3-17$$

An increase in $c_{i_k^*}$ may increase, but not decrease, the probability measure of the intersection (3-3-17) and may therefore increase, but not decrease, the probability measure of event (3-3-15).

$$\text{Hence, } \frac{\partial \Pr(\psi_j^*=1 \mid w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_k^*}} \geq 0 \quad 3-3-18$$

for all $k \geq j+1$, all $j = 1, \dots, j^*-1$.

If $k \leq j$, an increase in $c_{i_k}^*$ influences the event

$$\bigcap_{n=1}^{j-1} \{p_{1n}^{\min} - p_{1j}^{\min} > \sum_{m=n+1}^j c_{i_m}^*\} \quad 3-3-19$$

An increase in $c_{i_k}^*$ may decrease, but not increase, the probability measure of the intersection (3-3-19) and may therefore decrease, but not increase, the probability measure of event (3-3-15). Hence

$$\frac{\partial \Pr(\psi_j^*=1 | w, c_{i_1}^*, \dots, c_{i_{j^*}}^*)}{\partial c_{i_k}^*} \leq 0 \quad 3-3-20$$

for all $k \leq j$, all $j = 2, \dots, j^*$.

Combining (3-3-16), (3-3-18) and (3-3-20) gives the desired result.

Q.E.D.

Using Proposition 3-3-3, it is easy to show the ex ante expected length of search cannot be increased by an increase in any of the marginal financial costs of search. The ex ante expected search length is

$$E[j | w, c_{i_1}^*, \dots, c_{i_{j^*}}^*] = \sum_{j=1}^{j^*} j \cdot \Pr(\psi_j^*=1 | w, c_{i_1}^*, \dots, c_{i_{j^*}}^*) \quad 3-3-21$$

Proposition 3-3-3 shows that an increase in the k th marginal financial search cost ($k \neq 1$) may increase the probabilities in (3-3-21) for $j < k$ and may decrease the probabilities for $j \geq k$. Since (3-3-21) is a sum of these probabilities weighted by j , increasing from 1 to j^* , an increase in $c_{i_k}^*$ may decrease, but not increase, the ex ante expected search length.

Proposition 3-3-4:

$$\frac{\partial E[j | w, c_{i_1^*}, \dots, c_{i_j^*}]}{\partial c_{i_k^*}} \begin{cases} = 0, & \text{if } k = 1 \\ \leq 0, & \text{if } k = 2, \dots, j^* \end{cases}$$

Proof:

From (3-3-21),

$$\frac{\partial E[j | w, c_{i_1^*}, \dots, c_{i_{j^*}^*}]}{\partial c_{i_k^*}} = \sum_{j=1}^{j^*} j \cdot \frac{\partial \Pr(\psi_j^*=1 | w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_k^*}} \quad 3-3-22$$

$$= \sum_{j=1}^{k-1} j \cdot \frac{\partial \Pr(\psi_j^*=1 | w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_k^*}} + \sum_{j=k}^{j^*} j \cdot \frac{\partial \Pr(\psi_j^*=1 | w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_k^*}} \quad 3-3-23$$

where, by Proposition 3-3-3, the first summation in (3-3-23) is non-negative and the second summation is non-positive. From (3-3-23),

$$\frac{\partial E[j | w, c_{i_1^*}, \dots, c_{i_{j^*}^*}]}{\partial c_{i_k^*}} \leq k \cdot \sum_{j=1}^{j^*} \frac{\partial \Pr(\psi_j^*=1 | w, c_{i_1^*}, \dots, c_{i_{j^*}^*})}{\partial c_{i_k^*}} = 0 \quad 3-3-24$$

since the sum of all the components of the p.m.f. of search lengths is always unity. If $k = 1$ in particular, Proposition 3-3-3 shows (3-3-22) is zero.

Q.E.D.

It is important to realise that the results of Propositions 3-3-3 and 3-3-4 are valid only for small changes in marginal financial search costs. A large change in a marginal financial search cost could cause a change in the optimal search path and the results of these propositions are derived with the implicit assumption that no such change in the optimal path occurs.

This section has examined the manner in which financial search costs influence a searcher's ex ante decisions about the length of search, the physical method of search, the sellers contacted and the order in which they are contacted. Similar ex ante decisions on these variables will be made after each observation on p_1 up to the point where search is halted. The effect of financial search costs and the search path chosen on the optimal sequential search rule is taken up in Section 3-7.

The next section isolates the directional influences of transaction costs.

SECTION 3-4: TRANSACTION COSTS AND THE OPTIMAL SEARCH PATH

This section isolates the influences of transaction costs on the direction of search. The analysis proceeds in a fashion similar to that of Section 3-3. Influences due to financial search costs and searcher's knowledge of sellers' relative pricing behaviour are assumed not to exist. A search length j is chosen arbitrarily and the search path of length j and least expenditure on commodity 1 is found to be characterised by a monotonic increasing sequence of the smallest j transaction costs. The optimal search path is then located by finding the ex ante expected search length j^* for which the ex ante expected total expenditure on commodity 1 is the global minimum of the local minima for each value of $j \geq 1$.

Transaction costs may take many forms. Governments commonly impose stamp duties on certain types of transactions. A buyer may face a delivery charge for his unit of commodity 1. Alternatively, the buyer's transaction cost may simply represent his cost in returning to his home from the location of the seller from whom he purchased his

unit of commodity 1. In this case the search path chosen and, therefore, the marginal financial search costs incurred, will be chosen with some regard to lessening the transaction cost incurred.

The directional influences exerted by transaction costs will not be the same as the directional influences exerted by financial search costs because of the differences in the nature of these two types of cost. Two of the more important differences are:

(i) a searcher incurs only one transaction cost while he may incur many marginal financial search costs.

(ii) a transaction cost is incurred only at the time a purchase is made, but marginal financial search costs are incurred incrementally at the end of each of the periods over which search persists.

These two differences mean that, in search models where the demand for commodity 1 is fixed, such as the present model, transaction costs simply form a second component of a 'net price', the first component of which is an observed selling price for commodity 1. In the search models presented in Chapters 5 and 6, where demand for commodity 1 is free to be continuously variable w.r.t. prices and wealth, it is seen that transaction costs of the type considered in this thesis cannot be incorporated as part of a 'net price'. However, a sales tax which levies a fixed amount for every unit of commodity 1 purchased could still be incorporated as part of a 'net price' where demand is continuously variable w.r.t. prices and wealth.

In this section, directional influences due to searcher's knowledge of sellers' relative pricing behaviour are assumed not to exist by again invoking assumption (3-3-2). The ex ante p.m.f. of search lengths and the ex ante expected total expenditure on commodity 1

for a search path i_1, \dots, i_j are

$$\Pr(\psi_k^*=1 | w, c_{i_1}, \dots, c_{i_j}, t_{i_1}, \dots, t_{i_j}) \quad 3-4-1$$

and

$$ETE(i_1, \dots, i_j) = E\left[\sum_{k=1}^j \{(p_1+t)_k^{\min} + \sum_{m=1}^k c_{i_m}\}, \psi_k^* | w\right] \quad 3-4-2$$

In Section 3-3 directional influences due to transaction costs were removed by assumption (3-3-1) which assumed all transaction costs to be zero. However, no directional influence on search will be exerted by non-zero transaction costs if they are the same for each seller. The optimal search path will then be determined entirely by the directional influences of marginal financial search costs. The following proposition shows that, if all transaction costs are equal, the ex ante optimal search path is exactly the same as the marginal financial search cost determined optimal search path derived in Proposition 3-3-2.

Proposition 3-4-1:

If $t_{i_j} = t$, a constant, for all i_j , the optimal search path i_1^*, \dots, i_j^* is given by Proposition 3-3-2.

Proof:

If $t_{i_j} = t$ for all i_j , the ex ante expected total expenditure on commodity 1, (3-4-2), for any search path i_1, \dots, i_j is

$$ETE(i_1, \dots, i_j) = E\left[\sum_{k=1}^j \{p_{1k}^{\min} + \sum_{m=1}^k c_{i_m}\}, \psi_k^* | w\right] + t \quad 3-4-3$$

The ex ante probability of a search length of k , $1 \leq k \leq j$, is

$$\Pr(\psi_k^*=1 | w, c_{i_1}, \dots, c_{i_j}, t) = \Pr(TE_k < TE_n, \text{ for all } n \neq k, 1 \leq n \leq j | w, c_{i_1}, \dots, c_{i_j}, t) \quad 3-4-4$$

$$= \Pr\left(\bigcap_{\substack{n=1 \\ n \neq k}}^j \{p_{1k}^{\min} + \sum_{m=1}^k c_{i_m} + t < p_{1n}^{\min} + \sum_{m=1}^n c_{i_m} + t\} \mid w\right) \quad 3-4-5$$

$$= \Pr\left(\bigcap_{\substack{n=1 \\ n \neq k}}^j \{p_{1k}^{\min} + \sum_{m=1}^k c_{i_m} < p_{1n}^{\min} + \sum_{m=1}^n c_{i_m}\} \mid w\right) \quad 3-4-6$$

Comparison of (3-4-6) and (3-3-12) shows

$$\Pr(\psi_k^*=1 \mid w, c_{i_1}, \dots, c_{i_j}, t) = \Pr(\psi_k^*=1 \mid w, c_{i_1}, \dots, c_{i_j}) \quad 3-4-7$$

for all $k = 1, \dots, j$. The problem of selecting the search path for which (3-4-3) is minimised is therefore identical to the problem of selecting the search path for which (3-3-4) is minimised. The solution to this problem is provided by Proposition 3-3-2.

Q.E.D.

For the remainder of this section it is assumed that directional influences due to marginal financial search costs do not exist so that directional influences due to transaction costs may be examined in isolation.

Assumption: The marginal financial cost of an observation on p_1 is the same for each seller ie. $c_{i_j} = c$, a constant, for all i_j .

3-4-8

The ex ante p.m.f. of search lengths and the ex ante expected total expenditure on commodity 1, (3-4-1) and (3-4-2), subject to assumptions (3-3-2) and (3-4-8), for a search path i_1, \dots, i_j are

$$\Pr(\psi_k^*=1 \mid w, c, t_{i_1}, \dots, t_{i_j}) \quad 3-4-9$$

and

$$\text{ETE}(i_1, \dots, i_j) = E \left[\sum_{k=1}^j \{(p_1 + t)_k^{\min} + kc\} \cdot \psi_k^* \mid w \right] \quad 3-4-10$$

As in the case of marginal financial search costs, it is intuitive that, in the absence of other cost or benefit factors distinguishing one seller from another, the searcher will direct his search initially towards those sellers with the smallest transaction costs. This intuition is formalised in Proposition 3-4-2.

Proposition 3-4-2;

If $c_{i_j} = c > 0$ for all i_j , the search path of length $j \geq 1$ for which the ex ante expected total expenditure on commodity 1 is minimised is i_1^1, \dots, i_j^1 where $t_{i_m^1} = t_{(m)}$ for all $m = 1, \dots, j$.

Proof:

The reasoning involved in this proof is similar to that used in proving Proposition 3-3-1. (3-4-10) may be written in forms comparable to (3-3-5) and (3-3-6) of Proposition 3-3-1.

$$ETE(i_1, \dots, i_j) = \sum_{k=1}^j \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_k^* \{ (p_1 + t)_k^{\min} + kc \} \cdot f_g(p_1^1, \dots, p_1^j) dp_1^1 \dots dp_1^j$$

3-4-11

$$= \int_{p_1^L}^{p_1^U} \min\{p_1^1 + t_{i_1^1} + c, \int_{p_1^L}^{p_1^U} \min\{\min(p_1^1 + t_{i_1^1}, p_1^2 + t_{i_2^1}) + 2c, \int_{p_1^L}^{p_1^U} \min\{$$

$$\min(p_1^1 + t_{i_1^1}, p_1^2 + t_{i_2^1}, p_1^3 + t_{i_3^1}) + 3c, \dots\} f_g(p_1^3 | y_2) dp_1^3 \} f_g(p_1^2 | y_1) dp_1^2 \} f_g(p_1^1) dp_1^1$$

3-4-12

The problem is to select the order and values of the t_{i_m} for which (3-4-12) is minimised. Since t_{i_1} is a component of the total expenditures expected at each of the stages of search 1 to j,

$$\text{ETE}(i_1, \dots, i_j) \geq \int_{p_1^L}^{p_1^U} \min\{p_1^{1+t}(1) + c, \int_{p_1^L}^{p_1^U} \min\{\min(p_1^{1+t}(1), p_1^{2+t}i_2) + 2c,$$

$$\int_{p_1^L}^{p_1^U} \min\{\min(p_1^{1+t}(1), p_1^{2+t}i_2, p_1^{3+t}i_3) + 3c, \dots\} f_g(p_1^3|y_2) dp_1^3 \} f_g(p_1^2|y_1) dp_1^2 \} \\ f_g(p_1^1) dp_1^1 \quad 3-4-13$$

$$\geq \int_{p_1^L}^{p_1^U} \min\{p_1^{1+t}(1) + c, \int_{p_1^L}^{p_1^U} \min\{\min(p_1^{1+t}(1), p_1^{2+t}(2)) + 2c, \int_{p_1^L}^{p_1^U} \min\{ \\ \min(p_1^{1+t}(1), p_1^{2+t}(2), p_1^{3+t}i_3) + 3c, \dots\} f_g(p_1^3|y_2) dp_1^3 \} f_g(p_1^2|y_1) dp_1^2 \} \\ f_g(p_1^1) dp_1^1 \quad 3-4-14$$

$\geq \dots$

$$\geq \text{ETE}(i_1^*, \dots, i_j^*) \quad 3-4-15$$

for any search path i_1, \dots, i_j .

Q.E.D.

The characterisation of the optimal search path $i_1^*, \dots, i_{j^*}^*$ in terms of transaction costs follows immediately from Proposition 3-4-2.

Proposition 3-4-3:

The optimal search path $i_1^*, \dots, i_{j^*}^*$ is such that

$$(i) \quad t_{i_m^*} = t_{(m)} \text{ for all } m = 1, \dots, j^*$$

$$(ii) \quad \text{ETE}(i_1^*, \dots, i_{j^*}^*) \leq \text{ETE}(i_1, \dots, i_j)$$

for any search path i_1, \dots, i_j .

Proof:

(ii) above follows from definition (3-2-8). The optimal search path minimises the ex ante expected expenditure on commodity 1. Hence Proposition 3-4-2 may be applied to establish (i) above.

Q.E.D.

Increasing a transaction cost, like increasing a marginal financial search cost, is an increase in a disincentive for search and may be expected to decrease the ex ante expected length of search. Results for transaction costs akin to Propositions 3-3-3 and 3-3-4 for marginal financial search costs can be derived. So long as not all transaction costs associated with a search path are equal, the results are that an increase in the k th seller's transaction cost t_{i_k} may increase (cannot decrease) the ex ante probability of the search stopping after $j < k$ observations on p_1 and may decrease (cannot increase) the ex ante probability of the search stopping after $j \geq k$ observations on p_1 . However, if all transaction costs associated with a search path are equal, and are all altered by the same amount, the optimal search path will be unchanged. This may be seen from (3-4-6) of Proposition 3-4-1 which shows the ex ante p.m.f. of search lengths is independent of the value of equal transaction costs.

In Section 3-7, following an optimal search path given by Proposition 3-4-3 or Proposition 3-3-2 is shown to result in the coincidence of myopic and optimal sequential search behaviours.

The next section isolates and examines the last of the three factors mentioned as influencing the direction of search - searcher's knowledge of sellers' relative pricing behaviour.

SECTION 3-5: SELLERS' RELATIVE PRICING BEHAVIOURS AND THE OPTIMAL SEARCH PATH.

In this section directional influences on search due to financial search costs and transaction costs are assumed not to exist in order to isolate directional influences due to searcher's knowledge of sellers' relative pricing behaviour. Assumptions (3-4-8) and (3-5-1) are therefore applied throughout the whole of this section. Assumption: Transaction costs are equal for all sellers ie.

$$t_{i_j} = t, \text{ a constant, for all } i_j. \quad 3-5-1$$

The analysis of Section 3-3 was facilitated by assuming directional effects of transaction costs to be non-existent by assuming $t_{i_j} = 0$ for all i_j (assumption (3-3-1)). In fact, all that is required to ensure transaction costs have no directional effect is the weaker assumption just made, assumption (3-5-1). Proof of this is contained in Proposition 3-4-1 which shows transaction costs have no directional effects if they are all equal. Thus there is no need to assume all transaction costs are equal to zero in particular.

Since the searcher cannot distinguish any differences in cost or benefit factors between sellers other than those due to differences in relative pricing behaviour, it is intuitive that the searcher will initially direct his search towards those sellers he considers to be most likely to offer him relatively low selling prices. The formalisation of this intuition is presented in Propositions 3-5-3 and 3-5-4. The optimal search path is presented in Proposition 3-5-5. These propositions are dependent on Lemmas 3-5-1 and 3-5-2 which provide a description of how the selling price expected from the seller with index i_j will vary with a change in w_{i_j} .

In Section 1-4, w was defined as a vector of parameters conditioning the p.d.f. of selling prices $f(p_1|w)$. In the analysis prior to this section, assumption (3-3-2) has always required the searcher to regard all sellers as quoting selling prices from the same p.d.f. $f(p_1|w)$. At least some of the components of w are uncertain and the searcher is able to refine his prior beliefs about the likelihoods of different values of these unknown components by Bayesian pre-posterior updating of his prior p.d.f. $g(w)$. This makes him better informed about the likelihoods of receiving lower quotations from sellers not yet contacted.

To describe how differences in relative pricing behaviours direct search, assumption (3-3-2) is revoked. Instead it is assumed that the searcher regards each seller as quoting from his own p.d.f. of selling prices $f(p_1|w_{i_j})$. Since each seller is assumed passive in that he offers only the one selling price to all searchers who contact him, the p.d.f. $f(p_1|w_{i_j})$ describes the searcher's beliefs about the relative likelihoods of the possible values of the quotation from the seller with index i_j . Because only one selling price is offered by any one seller, there is no gain to be had from soliciting a second quotation from any one seller. It is also assumed that a quotation from one seller provides no information about the quotation offered by any other seller. Recall assumption (1-4-10) that all observations on p_1 are independently distributed. Independence between observations on p_1 and only one observation per seller make no gain possible from the searcher using pre-posterior updating to refine values of w ascribed to sellers further along his intended search path. Under these conditions the

ex ante p.m.f. of search lengths and the ex ante expected total expenditure on commodity 1 for a search path i_1, \dots, i_j are

$$\Pr(\psi_k^*=1|c,t,w_{i_1},\dots,w_{i_j}) \text{ for all } k = 1, \dots, j \quad 3-5-3$$

and

$$\text{ETE}(i_1, \dots, i_j) = E\left[\sum_{k=1}^j \{p_{1k}^{\min} + kc\} \psi_k^* | w_{i_1}, \dots, w_{i_j}\right] + t \quad 3-5-4$$

The choice of search path clearly depends upon the manner in which the p.d.f. $f(p_1|w_{i_j})$ depends upon the value of w_{i_j} . A large variety of such dependencies exist. The two particular choices examined in this section are that an increase in the value of w_{i_j} does not decrease, or does not increase, the probability of observing a quotation below a particular value of p_1 , for all values of $p_1 \in [p_1^L, p_1^U]$.

$$\text{ie, either } \frac{\partial F(p_1|w_{i_j})}{\partial w_{i_j}} \leq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-5-5$$

$$\text{or } \frac{\partial F(p_1|w_{i_j})}{\partial w_{i_j}} \geq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-5-6$$

Figure 3-5-1 illustrates the meaning of (3-5-5). Note that F is the cumulative p.d.f. of p_1 . The consequent changes in $f(p_1|w_{i_j})$ for (3-5-5) are illustrated in Figure 3-5-2. Figure 3-5-3 is a diagram of the function $\partial f(p_1^j|w_{i_j})/\partial w_{i_j}$, showing the relationship between the changes in $F(p_1^j|w_{i_j})$ caused by a change in the value of w_{i_j} to the changes in the ordinates of $f(p_1^j|w_{i_j})$.

An examination of Figure 3-5-2 should provide the intuition that an increase in w_{i_j} will cause an increase in $E[p_1|w_{i_j}]$ when (3-5-5) applies and, similarly, a decrease in $E[p_1|w_{i_j}]$ when (3-5-6) applies.

Proposition 3-5-3 formalises this intuition. Continuing this intuition, a searcher will prefer to solicit a quotation from the seller to whom he has ascribed the most favourable value of w ($w_{(1)}$ when (3-5-5) applies and $\bar{w}_{(1)}$ when (3-5-6) applies) since the quotation from this seller is expected to be smallest. This is the result of Proposition 3-5-4.

It is mathematically convenient to assume the p.d.f. of selling prices is differentiable w.r.t. w .

Assumption: $f(p_1|w)$ is differentiable w.r.t. w . 3-5-7

An example illustrating (3-5-7) and (3-5-6) is now given using the cumulative exponential distribution. The p.d.f. $f(p_1|w)$ for the exponential distribution is

$$f(p_1|w) = \begin{cases} 0, & p_1 \leq 0 \\ w \cdot e^{-wp_1}, & p_1 > 0 \end{cases} \quad \text{where } w > 0 \quad 3-5-8$$

The cumulative density function (c.d.f.), $F(p_1|w)$, is

$$F(p_1|w) = \begin{cases} 0, & p_1 \leq 0 \\ 1 - e^{-wp_1}, & p_1 > 0 \end{cases} \quad 3-5-9$$

Hence, $\frac{\partial F(p_1|w)}{\partial w} = p_1 \cdot e^{-wp_1} > 0$ for all $p_1 > 0$ 3-5-10

A point convenient to note now, and useful in understanding Lemma 3-5-1, is that (3-5-10) implies that $\frac{\partial f(p_1|w)}{\partial w}$ must initially be positive and later negative ie.

$$\frac{\partial f(p_1|w)}{\partial w} = (1 - wp_1)e^{-wp_1} \quad 3-5-11$$

Figure 3-5-1

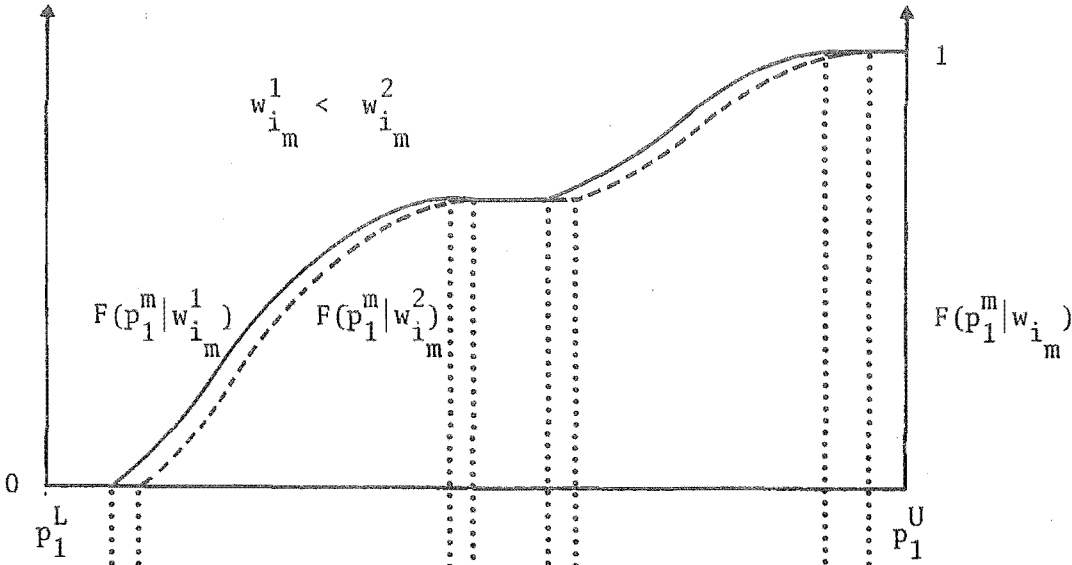


Figure 3-5-2

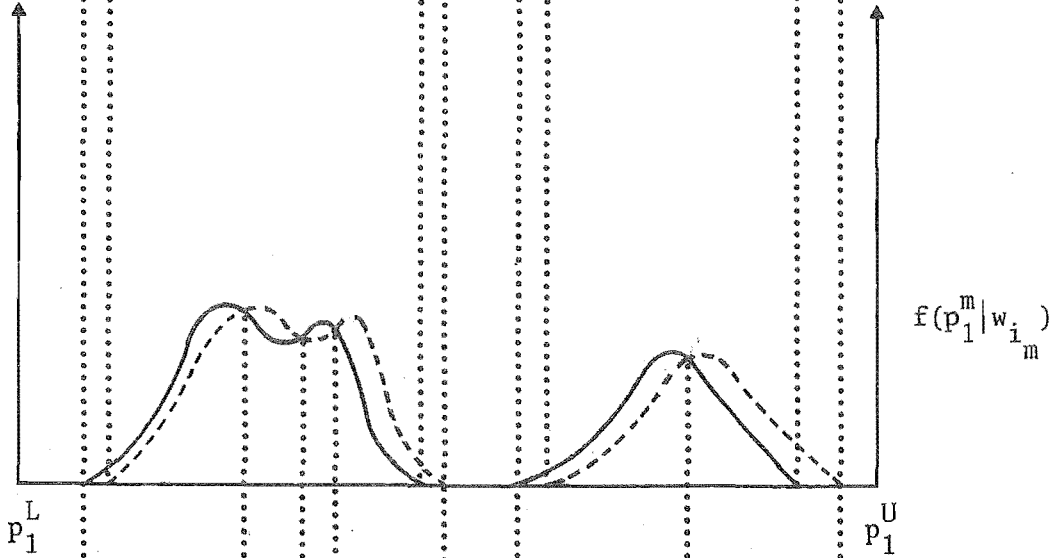
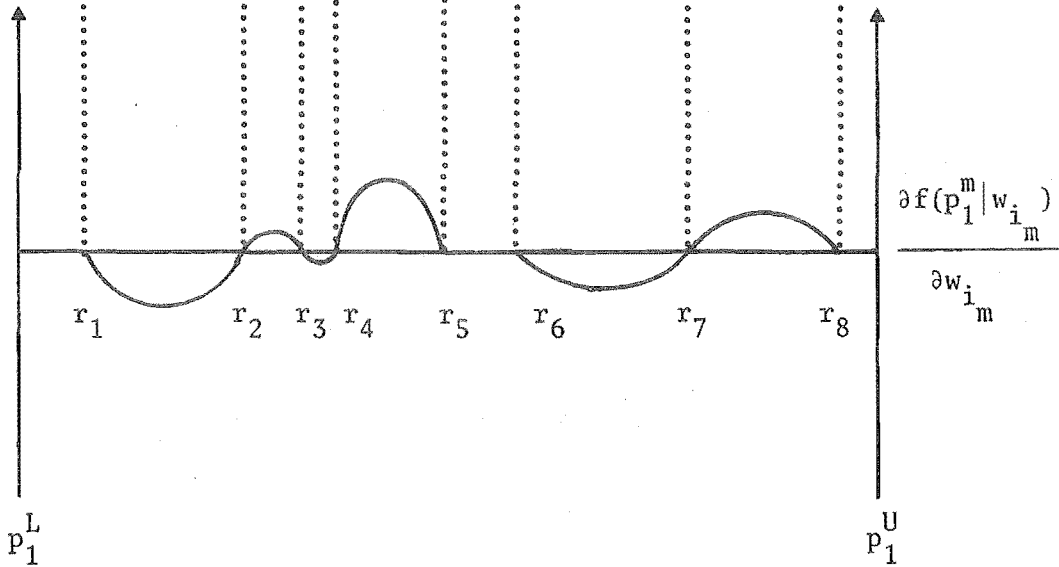


Figure 3-5-3



$$\text{so } \frac{\partial f(p_1|w)}{\partial w} \begin{cases} > 0, 0 < p_1 < 1/w \\ = 0, p_1 = 1/w \\ < 0, p_1 > 1/w \end{cases} \quad 3-5-12$$

This sign property is true for any p.d.f. $f(p_1|w)$ that is differentiable w.r.t. w and satisfies (3-5-6). The reverse sign property is true if the p.d.f. satisfies (3-5-5).

Lemmas 3-5-1 and 3-5-2 are now presented to describe how expectations taken w.r.t. $f(p_1|w)$ are changed by changes in w .

Lemma 3-5-1:

$$\int_{p_1^L}^{p_1^U} g(p_1^m) \frac{\partial f(p_1^m|w_{i_m})}{\partial w_{i_m}} dp_1^m \begin{cases} \geq 0, \text{ if } \frac{\partial g}{\partial p_1^m} \geq 0 \text{ and } \frac{\partial F(p_1^m|w_{i_m})}{\partial w_{i_m}} \leq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \\ \text{or if } \frac{\partial g}{\partial p_1^m} \leq 0 \text{ and } \frac{\partial F(p_1^m|w_{i_m})}{\partial w_{i_m}} \geq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \\ \leq 0, \text{ if } \frac{\partial g}{\partial p_1^m} \geq 0 \text{ and } \frac{\partial F(p_1^m|w_{i_m})}{\partial w_{i_m}} \geq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \\ \text{or if } \frac{\partial g}{\partial p_1^m} \leq 0 \text{ and } \frac{\partial F(p_1^m|w_{i_m})}{\partial w_{i_m}} \leq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \end{cases}$$

Proof:

$f(p_1^m|w_{i_m})$ is a proper p.d.f. so

$$\int_{p_1^L}^{p_1^U} f(p_1^m|w_{i_m}) dp_1^m = 1$$

3-5-13

Therefore,

$$\frac{\partial}{\partial w_{i_m}} \int_{p_1^L}^{p_1^U} f(p_1^m | w_{i_m}) dp_1^m = \int_{p_1^L}^{p_1^U} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m = 0 \quad 3-5-14$$

$$\text{Let } R^+ = \{p_1^m | \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} > 0\}, R^- = \{p_1^m | \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} < 0\} \quad 3-5-15$$

From (3-5-15),

$$R^+ \cap R^- = \emptyset \quad 3-5-16$$

and

$$\int_{R^-} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m + \int_{R^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m = 0 \quad 3-5-17$$

$$\text{The equation } \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} = 0 \quad 3-5-18$$

may have an infinite number of solutions since it is possible that $f(p_1^m | w_{i_m}) = 0$ for values of p_1^m contained in some continuous subset of $[p_1^L, p_1^U]$. Such a case is illustrated by the intervals $[p_1^L, r_1)$, (r_5, r_6) and $(r_8, p_1^U]$ in Figure 3-5-3. The set of such solutions to (3-5-18) is denoted by

$$R_0 = \{p_1^m | f(p_1^m | w_{i_m}) = \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} = 0\} \quad 3-5-19$$

The remaining solutions to (3-5-18) are denoted by $r_1, \dots, r_n \notin R_0$

$$\text{where } p_1^L \leq r_1 < r_2 < \dots < r_n \leq p_1^U \quad 3-5-20$$

For example, see r_1, \dots, r_8 in Figure 3-5-3. The remainder of the proof is carried out for only the first of the four cases listed in the statement of the Lemma. The reasoning may be replicated to

prove the result for any of the other three cases.

$$\text{Suppose } \frac{\partial g}{\partial p_1^m} \geq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \quad 3-5-21$$

$$\text{and } \frac{\partial F(p_1^m | w_{i_m})}{\partial w_{i_m}} \leq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \quad 3-5-22$$

The next part of the proof details the construction of a sequence of paired sets $\{R_i^-, R_i^+\}_{i=1}^{k^*}$ where $k^* \geq 1$. The first set R_1^- is the set of all values of p_1^m in the longest continuous interval of p_1^m from p_1^L upwards for which $\partial f(p_1^m | w_{i_m}) / \partial w_{i_m} \leq 0$. The second set R_1^+ is the set of all values of p_1^m in the longest continuous interval of p_1^m which has a greatest lower bound of R_1^- and for which $\partial f(p_1^m | w_{i_m}) / \partial w_{i_m} \geq 0$. The third set R_2^- is the set of all values of p_1^m in the longest continuous interval of p_1^m which has a greatest lower bound of R_1^+ and for which $\partial f(p_1^m | w_{i_m}) / \partial w_{i_m} \leq 0$, and so on for R_2^+, \dots (3-5-22) requires

$$\int_{r_1}^{r_2} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m \leq 0 \quad 3-5-23$$

$$\text{If } \int_{r_2}^{r_3} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m > 0,$$

$$\text{define } R_1^- = \{p_1^m | r_1 \leq p_1^m < r_2\} \quad 3-5-24$$

$$\text{If } \int_{r_2}^{r_3} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m \leq 0,$$

$$\text{and if } \int_{r_3}^{r_4} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m > 0,$$

$$\text{define } R_1^- = \{p_1^m | r_1 \leq p_1^m < r_3\} \quad 3-5-25$$

Continue in this way until

$$\int_{r_k}^{r_{k+1}} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m > 0 \quad 3-5-26$$

Then $R_1^- = \{p_1^m | r_1 \leq p_1^m < r_k\}$ and one considers 3-5-27

$$\int_{r_{k+1}}^{r_{k+2}} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq 0$$

$$\text{If } \int_{r_{k+1}}^{r_{k+2}} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m < 0,$$

define $R_1^+ = \{p_1^m | r_k \leq p_1^m < r_{k+1}\}$ 3-5-28

$$\text{If } \int_{r_{k+1}}^{r_{k+2}} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq 0,$$

$$\text{consider } \int_{r_{k+2}}^{r_{k+3}} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq 0$$

$$\text{If } \int_{r_{k+2}}^{r_{k+3}} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m < 0,$$

define $R_1^+ = \{p_1^m | r_k \leq p_1^m < r_{k+2}\}$ 3-5-29

A sequence of sets $R_1^-, R_1^+, \dots, R_{k*}^-, R_{k*}^+$ may be built up in this manner. The two conditions (3-5-22) and that

$$F(p_1^U | w_{i_m}) = 1 \text{ for all } w_{i_m} \quad 3-5-30$$

are sufficient to guarantee firstly that at least one pair of sets R_i^-, R_i^+ will exist and, secondly, that they must exist in pairs in the order R_i^-, R_i^+ for all $i = 1, \dots, k^*$. It should also be clear from the method of construction that

$$R_i^+ \cap R_j^+ = R_i^- \cap R_j^- = \emptyset, \text{ for all } i \neq j, \text{ where } i, j = 1, \dots, k^* \quad 3-5-31$$

$$R_i^- \cap R_j^+ = \emptyset, \text{ for all } i, j = 1, \dots, k^* \quad 3-5-32$$

$$\bigcup_{i=1}^{k^*} (R_i^- \cup R_i^+) = [p_1^L, p_1^U] \cap R_0 \quad 3-5-33$$

$$\bigcup_{i=1}^{k^*} R_i^- = R^-, \quad \bigcup_{i=1}^{k^*} R_i^+ = R^+ \quad 3-5-34$$

The bounds of the sets R_i^- and R_i^+ will be denoted by r_i^- , r_i and r_i^+ where

$$R_i^- = \{p_1^m | r_i^- \leq p_1^m < r_i\}, \quad R_i^+ = \{p_1^m | r_i \leq p_1^m < r_i^+\} \quad 3-5-35$$

$$\text{Note that } r_i^+ = r_{i+1}^-, \quad r_1^- \geq p_1^L, \quad r_{k^*}^+ \leq p_1^U \quad 3-5-36$$

From (3-5-31), (3-5-32), (3-5-33), (3-5-34), (3-5-35) and (3-5-17),

$$\sum_{i=1}^{k^*} \left\{ \int_{r_i^-}^{r_i} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m + \int_{r_i}^{r_i^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \right\} = 0 \quad 3-5-37$$

$$\begin{aligned} \text{Similarly, } & \int_{p_1^L}^{p_1^U} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \\ = & \sum_{i=1}^{k^*} \left\{ \int_{r_i^-}^{r_i} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m + \int_{r_i}^{r_i^+} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \right\} \end{aligned} \quad 3-5-38$$

(3-5-38) must be shown to be non-negative. Consider the first term in the summation (3-5-38).

$$\begin{aligned} \int_{r_1^-}^{r_1} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m &= \int_{p_1^L}^{r_1} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad \text{by (3-5-36)} \\ &\geq g(r_1) \int_{p_1^L}^{r_1} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \end{aligned} \quad 3-5-39$$

since $g(r_1) \geq g(p_1^m)$ for all $p_1^m \in [p_1^L, r_1)$ by (3-5-21)

and $\frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} < 0$ for all $p_1^m \in [p_1^L, r_1)$ by (3-5-35).

$$\int_{r_1}^{r_1^+} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq g(r_1) \int_{r_1}^{r_1^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad 3-5-40$$

since $g(r_1) \leq g(p_1^m)$ and $\frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} > 0$ for all $p_1^m \in [r_1, r_1^+)$ by (3-5-21)

and (3-5-35). Combining (3-5-39) and (3-5-40) gives

$$\int_{p_1^L}^{r_1^+} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq g(r_1) \cdot \int_{p_1^L}^{r_1^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad 3-5-41$$

Continuing in this way for each of terms in the summation (3-5-38) shows

$$\int_{p_1^L}^{p_1^U} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq \sum_{i=1}^{k^*} g(r_i) \int_{r_i^-}^{r_i^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad 3-5-42$$

The first term in the summation (3-5-42) is

$$g(r_1) \int_{p_1^L}^{r_1^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq g(r_2) \int_{p_1^L}^{r_1^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad 3-5-43$$

since $g(r_1) \leq g(r_2)$ by (3-5-21) and

$$\int_{p_1^L}^{r_1^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m = \frac{\partial F(r_1^+ | w_{i_m})}{\partial w_{i_m}} \leq 0 \text{ by (3-5-22).}$$

Inserting (3-5-43) into (3-5-42) gives

$$\begin{aligned} \int_{p_1^L}^{p_1^U} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m &\geq \sum_{i=3}^{k^*} g(r_i) \int_{r_i^-}^{r_i^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \\ &\quad + g(r_2) \int_{p_1^L}^{r_2^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \end{aligned} \quad 3-5-44$$

$$g(r_2) \int_{p_1^L}^{r_2^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq g(r_3) \int_{p_1^L}^{r_2^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad 3-5-45$$

since $g(r_2) \leq g(r_3)$ by (3-5-21) and

$$\int_{p_1^L}^{r_2^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m = \frac{\partial F(r_2^+ | w_{i_m})}{\partial w_{i_m}} \leq 0 \text{ by (3-5-22)}$$

Inserting (3-5-45) into (3-5-44) gives

$$\begin{aligned} \int_{p_1^L}^{p_1^U} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m &\geq \sum_{i=4}^{k^*} g(r_i) \int_{r_i^-}^{r_i^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \\ &\quad + g(r_3) \int_{p_1^L}^{r_3^+} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \end{aligned} \quad 3-5-46$$

Continuing in this way shows that

$$\int_{p_1^L}^{p_1^U} g(p_1^m) \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \geq g(x_{k^*}) \int_{p_1^L}^{p_1^U} \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m = 0 \quad 3-5-47$$

by (3-5-37).

This proves the desired result for conditions (3-5-21) and (3-5-22). Varying these conditions and applying the same reasoning given in the above proof will give the desired results for the remaining three cases.

Q.E.D.

The reasoning in this proof is also used in Lemma 4-2-4 where it is assumed that $\frac{\partial F_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^m} < 0$ for all $p_1 \in [p_1^L, p_1^U]$ where $p_1^m \in \{p_1^1, \dots, p_1^j\}$. This similarity of the form of this assumption to assumption (3-5-5) is obvious. The above reasoning is applicable to any p.d.f. satisfying a condition of this kind and so the result of Lemma 3-5-1 is more general than a discussion of only search paths might seem to imply.

Lemma 3-5-2:

If $\frac{\partial g_1}{\partial p_1^m} \geq \frac{\partial g_2}{\partial p_1^m}$ for all $p_1^m \in [p_1^L, p_1^U]$, then

$$\int_{p_1^L}^{p_1^U} g_1(p_1^m) \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m \geq \int_{p_1^L}^{p_1^U} g_2(p_1^m) \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m$$

as $\frac{\partial F(p_1^m | w_{i_m})}{\partial w_{i_m}} \leq 0$ for all $p_1^m \in [p_1^L, p_1^U]$.

Proof:

$$\text{Let } g(p_1^m) = g_1(p_1^m) - g_2(p_1^m) \quad 3-5-48$$

$$\text{Since } \frac{\partial g_1}{\partial p_1^m} \geq \frac{\partial g_2}{\partial p_1^m} \text{ for all } p_1^m \in [p_1^L, p_1^U],$$

$$\frac{\partial g}{\partial p_1^m} \geq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \quad 3-5-49$$

By Lemma 3-5-1,

$$\int_{p_1^L}^{p_1^U} g(p_1^m) \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m = \int_{p_1^L}^{p_1^U} (g_1(p_1^m) - g_2(p_1^m)) \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} dp_1^m \geq 0 \quad 3-5-50$$

$$\text{as } \frac{\partial F(p_1^m | w_{i_m})}{\partial w_{i_m}} \geq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U].$$

Q.E.D.

The effect a change in w_{i_j} has on the selling price expected to be quoted by the seller with index i_j can immediately be determined from Lemma 3-5-1. The result is presented in Proposition 3-5-3.

Proposition 3-5-3:

$$\frac{\partial}{\partial w_{i_m}} (E[p_1^m | w_{i_m}]) \begin{cases} \geq 0, \text{ if } \frac{\partial F(p_1^m | w_{i_m})}{\partial w_{i_m}} \leq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \\ \leq 0, \text{ if } \frac{\partial F(p_1^m | w_{i_m})}{\partial w_{i_m}} \geq 0 \text{ for all } p_1^m \in [p_1^L, p_1^U] \end{cases}$$

Proof:

$$E[p_1^m | w_{i_m}] = \int_{p_1^L}^{p_1^U} p_1^m \cdot f(p_1^m | w_{i_m}) dp_1^m \quad 3-5-51$$

Therefore,

$$\frac{\partial}{\partial w_{i_m}} (E[p_1^m | w_{i_m}]) = \int_{p_1^L}^{p_1^U} p_1^m \cdot \frac{\partial f(p_1^m | w_{i_m})}{\partial w_{i_m}} \cdot dp_1^m \quad 3-5-52$$

Since $\frac{\partial}{\partial p_1^m}(p_1^m) = 1 > 0$, the desired result follows immediately from

Lemma 3-5-1.

Q.E.D.

If (3-5-5) applies ie. $\frac{\partial F(p_1 | w)}{\partial w} \leq 0$ for all $p_1 \in [p_1^L, p_1^U]$,

Proposition 3-5-3 shows the smallest selling price is expected from the seller with index i_j where $w_{i_j} = w_{(1)}$. The next smallest selling price is expected from the seller with index i_k where $w_{i_k} = w_{(2)}$ and so on. The intuition mentioned at the start of this section is that, when (3-5-5) applies, the searcher will direct his search so that

$w_{i_1} = w_{(1)}$, $w_{i_2} = w_{(2)}$ and so on.

If (3-5-6) applies ie. $\frac{\partial F(p_1 | w)}{\partial w} \geq 0$ for all $p_1 \in [p_1^L, p_1^U]$,

similar intuitive reasoning suggests the searcher will direct his search so that $w_{i_1} = \bar{w}_{(1)}$, $w_{i_2} = \bar{w}_{(2)}$ and so on.

Proposition 3-5-4 completes the formalisation of the intuition.

Proposition 3-5-4:

The search path of length $j \geq 1$ for which the ex ante expected total expenditure on commodity 1 is minimised is i'_1, \dots, i'_j where

- (a) $w_{i'_m} = w_{(m)}$ for all $m = 1, \dots, j$ if $\frac{\partial F(p_1 | w)}{\partial w} \leq 0$ for all $p_1 \in [p_1^L, p_1^U]$
- (b) $w_{i'_m} = \bar{w}_{(m)}$ for all $m = 1, \dots, j$ if $\frac{\partial F(p_1 | w)}{\partial w} \geq 0$ for all $p_1 \in [p_1^L, p_1^U]$.

Proof:

The ex ante expected total expenditure on commodity 1 for a search path i_1, \dots, i_j is given by (3-5-4) as

$$\begin{aligned} \text{ETE}(i_1, \dots, i_j) &= t + E\left[\sum_{k=1}^j \{p_{1k}^{\min} + kc\} \cdot \psi_k^* | w_{i_1}, \dots, w_{i_j}\right] \\ &= t + \sum_{k=1}^j \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_k^* \{p_{1k}^{\min} + kc\} \cdot f(p_1^1, \dots, p_1^k | w_{i_1}, \dots, w_{i_k}) dp_1^1 \dots dp_1^k \end{aligned} \quad 3-5-53$$

p_1^1, \dots, p_1^j are assumed to be independently distributed by (1-4-10)

$$\text{so } f(p_1^1, \dots, p_1^k | w_{i_1}, \dots, w_{i_k}) = \prod_{m=1}^k f(p_1^m | w_{i_m}) \text{ for all } k = 1, \dots, j. \quad 3-5-54$$

Substituting (3-5-54) into (3-5-53) gives

$$\text{ETE}(i_1, \dots, i_j) = t + \sum_{k=1}^j \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_k^* \{p_{1k}^{\min} + kc\} \prod_{m=1}^k f(p_1^m | w_{i_m}) dp_1^m \quad 3-5-55$$

Rewriting (3-5-55) in the form of (3-3-6) gives

$$\begin{aligned} \text{ETE}(i_1, \dots, i_j) &= t + \int_{p_1^L}^{p_1^U} \min\{p_1^1 + c, \int_{p_1^L}^{p_1^U} \min\{\min(p_1^1, p_1^2) + 2c, \int_{p_1^L}^{p_1^U} \min\{\min(p_1^1, p_1^2, p_1^3) \\ &\quad + 3c, \dots\} f(p_1^3 | w_{i_3}) dp_1^3\} f(p_1^2 | w_{i_2}) dp_1^2\} f(p_1^1 | w_{i_1}) dp_1^1 \end{aligned} \quad 3-5-56$$

$$\begin{aligned} &= t + \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \min\{p_1^1 + c, \min(p_1^1, p_1^2) + 2c, \min(p_1^1, p_1^2, p_1^3) + 3c, \dots \\ &\quad \dots, \min(p_1^1, \dots, p_1^j) + jc\} \prod_{m=1}^j f(p_1^m | w_{i_m}) dp_1^m \end{aligned} \quad 3-5-57$$

$$= t + \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \min\{p_1^1 + c, p_1^2 + 2c, p_1^3 + 3c, \dots, p_1^j + jc\} \prod_{m=1}^j f(p_1^m | w_{i_m}) dp_1^m \quad 3-5-58$$

The rate of change of $ETE(i_1, \dots, i_j)$ w.r.t. w_{i_k} , for any $k = 1, \dots, j$ is

$$\frac{\partial ETE}{\partial w_{i_k}} = \int_{p_1^L}^{p_1^U} \left[\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \min\{p_1^1+c, p_1^2+2c, \dots, p_1^k+kc, \dots, p_1^j+jc\} \cdot \pi_{m \neq k} f(p_1^m | w_{i_m}) dp_1^m \right] \frac{\partial f(p_1^k | w_{i_k})}{\partial w_{i_k}} \cdot dp_1^k \quad 3-5-59$$

Clearly

$$\frac{\partial}{\partial p_1^k} \left[\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \min\{p_1^1+c, \dots, p_1^k+kc, \dots, p_1^j+jc\} \cdot \pi_{m \neq k} f(p_1^m | w_{i_m}) dp_1^m \right] \geq 0 \quad 3-5-60$$

so, from Lemma 3-5-1 and (3-5-59),

$$\frac{\partial ETE}{\partial w_{i_k}} \geq 0 \text{ for all } k = 1, \dots, j \text{ if } \frac{\partial F(p_1 | w)}{\partial w} \leq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-5-61$$

and

$$\frac{\partial ETE}{\partial w_{i_k}} \leq 0 \text{ for all } k = 1, \dots, j \text{ if } \frac{\partial F(p_1 | w)}{\partial w} \geq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-5-62$$

The remainder of the proof shows that the rate of change of ETE w.r.t. w_{i_1} is greater than the rate w.r.t. w_{i_2} and so on to w_{i_j} . Then, since $ETE(i_1, \dots, i_j)$ is most sensitive to the value of w_{i_1} , the searcher should place his most favoured value of w first on his search path i.e. he should choose $w_{i_1} = w_{(1)}$ if (3-5-61) is true or $w_{i_1} = \bar{w}_{(1)}$ if (3-5-62) is true.

$$\frac{\partial}{\partial p_1^k} \left[\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \min\{p_1^1+c, p_1^2+2c, \dots, p_1^k+kc, \dots, p_1^j+jc\} \cdot \pi_{m \neq k} f(p_1^m | w_{i_m}) dp_1^m \right]$$

$$\begin{aligned}
&= \frac{\partial}{\partial p_1^k} \int_{p_1^L}^{p_1^k + (k-1)c} (p_1^1 + c) \cdot \{ (1-F(p_1^1 - c | w_{i_2})) \dots (1-F(p_1^1 - (k-2)c | w_{i_{k-1}})) (1-F(p_1^1 - kc | w_{i_{k+1}})) \dots \\
&\quad \dots (1-F(p_1^1 - (j-1)c | w_{i_j})) \} \cdot f(p_1^1 | w_{i_1}) dp_1^1 \\
&+ \int_{p_1^L}^{p_1^k + (k-2)c} (p_1^2 + 2c) \cdot \{ (1-F(p_1^2 + c | w_{i_1})) (1-F(p_1^2 - c | w_{i_3})) \dots (1-F(p_1^2 - (k-3)c | w_{i_{k-1}})) \cdot \\
&\quad (1-F(p_1^2 - (k-1)c | w_{i_{k+1}})) \dots (1-F(p_1^2 - (j-2)c | w_{i_j})) \} \cdot f(p_1^2 | w_{i_2}) dp_1^2 \\
&+ \dots \\
&+ (p_1^k + kc) \{ (1-F(p_1^k + (k-1)c | w_{i_1})) \dots (1-F(p_1^k + c | w_{i_{k-1}})) (1-F(p_1^k - c | w_{i_{k+1}})) \dots \\
&\quad \dots (1-F(p_1^k - (j-k)c | w_{i_j})) \} \cdot 1 \\
&+ \dots \\
&+ \int_{p_1^L}^{p_1^k - (j-k)c} (p_1^j + jc) \{ (1-F(p_1^j + (j-1)c | w_{i_1})) \dots (1-F(p_1^j + (j-k+1)c | w_{i_{k-1}})) \cdot \\
&\quad (1-F(p_1^j + (j-k-1)c | w_{i_{k+1}})) \dots (1-F(p_1^j + c | w_{i_{j-1}})) \} \cdot f(p_1^j | w_{i_j}) dp_1^j \quad 3-5-63 \\
&= (p_1^k + kc) (1-F(p_1^k + (k-2)c | w_{i_2})) \dots (1-F(p_1^k + (k-j)c | w_{i_j})) \cdot f(p_1^k + (k-1)c | w_{i_1}) \\
&+ (p_1^k + kc) (1-F(p_1^k + (k-1)c | w_{i_1})) \dots (1-F(p_1^k + (k-j)c | w_{i_j})) \cdot f(p_1^k + (k-2)c | w_{i_2}) \\
&+ \dots \\
&+ \{ (1-F(p_1^k + (k-1)c | w_{i_1})) \dots (1-F(p_1^k + c | w_{i_{k-1}})) (1-F(p_1^k - c | w_{i_{k+1}})) \dots \\
&\quad (1-F(p_1^k + (k-j)c | w_{i_j})) \}
\end{aligned}$$

$$\begin{aligned}
& - (p_1^k + kc) (1 - F(p_1^k + (k-2)c | w_{i_2})) \dots (1 - F(p_1^k + (k-j)c | w_{i_j})) \cdot f(p_1^k + (k-1)c | w_{i_1}) \\
& - \dots \\
& - (p_1^k + kc) (1 - F(p_1^k + (k-1)c | w_{i_1})) \dots (1 - F(p_1^k + (k-j-1)c | w_{i_{j-1}})) \cdot f(p_1^k + (k-j)c | w_{i_j}) \\
& + \dots \\
& + (p_1^k + kc) (1 - F(p_1^k + (k-1)c | w_{i_1})) \dots (1 - F(p_1^k + (k-j-1)c | w_{i_{j-1}})) \cdot f(p_1^k + (k-j)c | w_{i_j})
\end{aligned}$$

3-5-64

$$= \sum_{\substack{m=1 \\ m \neq k}}^j (1 - F(p_1^k + (k-m)c | w_{i_m}))$$

3-5-65

(3-5-65) is non-negative and non-increasing as k increases from 1 to j . Therefore Lemma 3-5-2 can be applied to give

$$\frac{\partial \text{ETE}}{\partial w_{i_1}} \geq \frac{\partial \text{ETE}}{\partial w_{i_2}} \geq \dots \geq \frac{\partial \text{ETE}}{\partial w_{i_j}} \geq 0 \text{ if } \frac{\partial F(p_1 | w)}{\partial w} \leq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-5-66$$

and

$$\frac{\partial \text{ETE}}{\partial w_{i_1}} \leq \frac{\partial \text{ETE}}{\partial w_{i_2}} \leq \dots \leq \frac{\partial \text{ETE}}{\partial w_{i_j}} \leq 0 \text{ if } \frac{\partial F(p_1 | w)}{\partial w} \geq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-5-67$$

(a) Suppose (3-5-61) applies. Since $\text{ETE}(i_1, \dots, i_j)$ decreases most rapidly if w_{i_1} is decreased, w_{i_1} should be chosen equal to $w_{(1)}$ to minimise $\text{ETE}(i_1, \dots, i_j)$ w.r.t. w_{i_1} . Then, since $\text{ETE}(i_1, \dots, i_j)$ decreases at the next most rapid rate if w_{i_2} is decreased, w_{i_2} should be chosen equal to $w_{(2)}$ to minimise $\text{ETE}(i_1, \dots, i_j)$ w.r.t. w_{i_1} and w_{i_2} . Continuing in this manner shows that, when (3-5-61) applies, the search path of length j which minimises $\text{ETE}(i_1, \dots, i_j)$ is $i_1^!, \dots, i_j^!$ where $w_{i_m} = w_{(m)}$ for all $m = 1, \dots, j$.

(b) Suppose (3-5-62) applies. A similar argument to that in (a) above shows the search path of length j which minimises $ETE(i_1, \dots, i_j)$ is i_1^*, \dots, i_j^* where $w_{i_m^*} = \bar{w}_{(m)}$ for all $m = 1, \dots, j$.

Q.E.D.

The characteristics of the optimal search path are immediate from definition (3-2-8) and the results of Proposition 3-5-4.

Proposition 3-5-5:

The optimal search path $i_1^*, \dots, i_{j^*}^*$ is such that

$$(i) \quad (a) \quad \text{if } \frac{\partial F(p_1 | w)}{\partial w} \leq 0 \text{ for all } p_1 \in [p_1^L, p_1^U], w_{i_m^*} = w_{(m)}$$

for all $m = 1, \dots, j^*$

$$(b) \quad \text{if } \frac{\partial F(p_1 | w)}{\partial w} \geq 0 \text{ for all } p_1 \in [p_1^L, p_1^U], w_{i_m^*} = \bar{w}_{(m)}$$

for all $m = 1, \dots, j^*$

$$(ii) \quad ETE(i_1^*, \dots, i_{j^*}^*) \leq ETE(i_1, \dots, i_j) \text{ for any search path } i_1, \dots, i_j.$$

Proof:

(ii) above follows from definition (3-2-8). The optimal search path minimises the ex ante expected total expenditure on commodity 1. Hence Proposition 3-5-4 may be applied to establish (i)(a) and (i)(b) above.

Q.E.D.

The recognition that a searcher's beliefs about sellers' relative pricing behaviour influence the direction of his search has resulted in a large, complex and persistently pervasive industry, the object of which is to influence these beliefs. In Section 3-6 the results of Sections 3-3, 3-4 and 3-5 are applied to understanding some of the strategies employed by modern advertisers in pursuance

of this end.

In Section 3-7 it is shown that following the optimal search path described by Proposition 3-5-5 results in the coincidence of myopic and optimal sequential search behaviour.

SECTION 3-6: ADVERTISING

Approximately one half of Stigler's seminal consumer search paper [50] is concerned with the relationship between search activity and advertising. Yet the search literature has chosen to largely ignore the relationship. This is directly attributable to the literature's lack of consideration of search paths. Advertising forces itself upon all economic agents of the modern world whether they like it or not. All agents' search behaviour is therefore modified by an information flow due to advertisements. The modification is firstly to the stock of information possessed by the individual and secondly to the individual's preferences. Advertisements need not be planned. Casual observation of other agents' behaviour may modify an agent's behaviour. Observing the next door neighbour's sports car may suggest that owning a sports car is desirable. This section is restricted to the much narrower topic of those aspects of advertising related to prices and information on sellers' characteristics.

Only three efforts have been made by authors other than Stigler to relate search activity and advertising to each other. Butter's advertising model [3] does combine the two, but only with difficulty. His model permits search if and only if consumers receive no sellers' advertisements and the conditions of search are such that consumers accept the first selling price quoted to them. Nelson's largely

empirical paper [34] contains a lengthy discussion on the manner in which sellers' advertising strategies are influenced by the nature of the goods they sell. No formal theory is presented but Nelson presents the following conclusions. If searchers are easily able to discern the qualities of a commodity, then sellers' advertisements will contain a higher level of factual information concerning the purchase of some of the commodity than otherwise. Conversely, sellers' advertisements for durable goods, infrequently purchased and possessing qualities accurately discernible only by use, will contain a lower level of fact and more irrelevant "information". Manning [28] discusses a seller who is able to exactly determine the number of searchers who will contact him within a certain period by varying his expenditure on advertising. Optimal advertising expenditure is related to sellers' tastes for privacy and income. On the assumption that the marginal utility of income is diminishing, Manning's model predicts that wealthier sellers "advertise less, quote lower prices and sell more quickly" [28, p. 316].

Not all advertising is directed by sellers towards buyers. Advertisements by potential buyers towards potential sellers is not discussed by any of Stigler, Nelson or Butters, although Stigler does recognise its existence. Every daily newspaper carries a column of "wanted" advertisements from potential buyers hoping to locate sellers other than those already known to them. Such advertising amounts to a search for sellers before, or concurrent with, a search for a low expenditure on the commodity. The motivation for identifying additional sellers is that it may provide a new search path of lower expected expenditure than the previously optimal search path. Recall a comment

from Section 3-3 that a finite change in a marginal financial search cost could cause a completely new search path to become optimal. The same effect could result from locating a previously unknown seller if the marginal cost of obtaining a quotation from this seller is smaller than the marginal cost of obtaining a quotation from a seller on the hitherto optimal search path. Similar changes could result if a newly identified seller's transaction cost is smaller than the transaction cost of a seller on the optimal search path, or if the new seller seems relatively likely to offer a low selling price. Of course, such a "pre-search" search will be carried out if and only if an agent expects its cost to be exceeded by the gain due to an improved search path becoming available.

Sellers' advertising is obviously more prevalent, more sophisticated and more widely broadcast than buyers' advertising. The object of a seller's advertising is to either direct as many searchers as possible towards him or else to direct them away from competing sellers. The strategies employed to these ends by an advertiser can be viewed as attempts to place himself at an earlier position on searchers' search paths. He can do this either by arguing that his offers are relatively advantageous to searchers or by arguing that his competitors' offers are relatively disadvantageous. The earlier is the advertiser's position on a searcher's path, the more likely it is that the searcher will actually ask the seller for a quotation and so give the seller a chance of making a sale. The strategies used by an advertising seller depend on the seller's assessment of his ability to compete against other sellers. In any

real market, sellers' competitive abilities vary widely and this is in part the reason for the wide variation observed in the informational content of sellers' advertisements. A feature common to all sellers' advertisements, however, is that they provide searchers with information on how to contact sellers. An advertisement may contain the seller's address, his telephone number, his hours of business and so forth. Some advertisements may be very specific eg. they may provide exact selling prices and transaction costs, terms of payment and perhaps offers of free delivery of goods purchased. The specificity of any advertisement will be partly determined by the competitive position of the seller. The preceding sections make it clear that competitiveness is not to be judged merely in terms of relative prices alone but in terms of relative total expenditures associated with each seller. Since it is unlikely that a seller's advertising of conditions of relatively high expenditure for a commodity will direct many searchers towards him, it is likely that advertisements with specific total expenditure information are issued by sellers in relatively competitive positions. The veracity of less specific advertising phrases such as "really great prices" are, therefore, subject to some doubt in most people's minds. That doubt matters is a consequence of the existence of search costs. Once a searcher is lured to a seller there is an additional cost incurred by the searcher, the next marginal search cost, if he does not make his purchase from this seller. Vague, or even deceptive, advertising can be regarded as an attempt by sellers to grasp this margin. The existence of legislation restricting the deceptiveness of advertisements is, in part, due to the recognition of this.

Sellers with locations more accessible to searchers than locations of lower priced sellers may still receive higher patronage. For instance, some large emporiums that have their own parking lots, and are in the midst of many sellers of other commodities, are able to charge higher prices than small, more isolated, suburban sellers with fewer customers. The advertisements from the sellers with favoured locations will presumably emphasise their locations and parking facilities and avoid emphasising their prices, except perhaps for the prices of a few specially selected commodities. These advertisements are directed at exploiting a trade off made by consumers between the benefits of purchasing at lower prices and the higher costs of patronising the lower priced sellers whose shop frontages are perhaps on busy clogged streets where little parking is available.

Whatever the wording, the underlying object of a seller's advertisement is to cause a searcher to reorder his search path so as to place the advertising seller in an earlier position on the path. If advertisements do cause a searcher to reorder his search path, it is because the new search path represents a lower expected total expenditure on the commodity and, therefore, a greater gain from search. Advertisements may, therefore, benefit both buyer and seller. For the searcher, any increase in the price of advertised commodities caused by sellers offsetting their advertising costs is offset by a lowering of their search costs. For the seller, the increase in revenue from higher patronage offsets his advertising costs.

The effect of advertising upon the length of search is indeterminate. If a very low priced seller reveals himself, a searcher

will find little gain in continuing his search past this seller. If advertisements reveal several additional sellers, a searcher may find it optimal to contact more sellers than previously. Whichever of these possibilities is dominant, however, the *mean total expenditure* on the searched for commodity should fall as the amount of truthful information provided to buyers by advertising increases.

SECTION 3-7: OPTIMAL SEARCH PATHS AND MYOPIC SEARCH RULES

In Section 2-1 it was noted that an assertion common in previous authors' sequential search models (eg. [1], [3], [11], [43], [52]) is that a myopic expenditure minimisation search rule of the form (2-1-1) is optimal. None of these models consider search paths and contain not the slightest notion of the direction of search. This section shows that the most logical justification for assuming optimality for a myopic search rule is to assume the searcher follows an optimal search path.

Section 2-7 showed that myopic sequential search rules are equivalent to optimal sequential search rules if and only if the maximum component of the sequence of utilities expected at subsequent stages of search is, after a certain length of search, always the utility expected at the very next stage of search. Assuming the conditions of Proposition 2-7-1 throughout this chapter ensures the equivalence of consumer utility maximisation and minimisation of total expenditure on commodity 1. These conditions cause the optimal sequential search rule to be equivalent to a myopic sequential search rule of the form (2-1-1) when the minimum component of the sequence of total expenditures on commodity 1 expected at subsequent stages of search is the expenditure expected at the next stage.

This section shows some conditions under which following an optimal search path results in such a sequence of expected expenditures (a submartingale). Myopic sequential search rules are thus shown to be optimal more frequently than it would appear from an examination of the above models, which make no attempt to justify their implicit assumption of their searchers' indifference to all possible search paths.

The optimal search paths derived in Propositions 3-3-2, 3-4-3 and 3-5-5 all create conditions under which the sequence of marginal expected total expenditures on commodity 1 is monotonic increasing, ie. a submartingale sequence. This is because the Propositions ensure that the marginal expected reduction in total expenditure from an additional observation on p_1 is non-increasing while also ensuring that the marginal cost of an additional observation on p_1 is non-decreasing. Once the marginal expected net change in total expenditure on commodity 1 becomes positive, therefore, it will remain positive for any subsequent observations. The searcher will always choose to stop searching at this point and, to locate this point, he need ever only look one stage ahead.

Propositions 3-7-1, 3-7-2 and 3-7-3 prove the optimality of myopic sequential expenditure minimisation search rules for the search conditions of Propositions 3-3-2, 3-4-3 and 3-5-5 respectively.

Proposition 3-7-1:

If $w_{i_j} = w$ and $t_{i_j} = t$ for all i_j , and if minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility, then the optimal sequential search rule ρ_s^* is myopic.

Proof:

$$\text{If } w_{i_j} = w \text{ for all } i_j \quad 3-7-1$$

$$\text{and if } t_{i_j} = t \text{ for all } i_j \quad 3-7-2$$

then the optimal search path is given by Proposition 3-3-2.

Therefore the sequence of marginal financial search costs incurred by following the optimal search path of Proposition 3-3-2 is monotonic increasing

$$c_{i_j}^* \leq c_{i_{j+1}}^* \quad \text{for all } j = 1, \dots, j^*-1 \quad 3-7-3$$

Since minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility, the optimal sequential rule is

$$\xi^* = (\xi_0^*, \xi_1^*, \dots, \xi_j^*, \dots) \text{ where } \xi_0^*(y_0) = S_0^* \equiv 0 \text{ and, from (2-6-2),}$$

$$\xi_j^*(y_j) = S_j^* = \begin{cases} 0, & TE_j(\delta^*) > E[W_{j+1}(\xi^*, \delta^*) | y_j] \\ 1, & TE_j(\delta^*) \leq E[W_{j+1}(\xi^*, \delta^*) | y_j] \end{cases} \quad 3-7-4$$

It must be shown that conditions (3-7-1), (3-7-2) and (3-7-3) are sufficient for (3-7-4) to be equivalent to the myopic rule where, by (2-7-44),

$$\xi_j^*(y_j) = S_j^* = \begin{cases} 0, & TE_j(\delta^*) > E[TE_{j+1}(\delta^*) | y_j] \\ 1, & TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \end{cases} \quad 3-7-5$$

By definition,

$$E[W_{j+1}(\xi^*, \delta^*) | y_j] = E[\min(TE_{j+1}(\delta^*), E[W_{j+2}(\xi^*, \delta^*) | y_{j+1}]) | y_j] \quad 3-7-6$$

$$= E[\min(TE_{j+1}(\delta^*), E[\min(TE_{j+2}(\delta^*), \dots) | y_{j+1}]) | y_j] \quad 3-7-7$$

Suppose that, after j observations on p_1 , the searcher expects that the total expenditure required on commodity 1 will be no smaller after $(j+1)$ observations on p_1 than at present ie,

$$TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \quad 3-7-8$$

Under conditions (3-7-1) and (3-7-2), (3-7-8) may be written as

$$p_{1j}^{\min} + \sum_{m=1}^j c_{i_m^*} + t \leq \int_{p_1^L}^{p_{1j}^{\min}} p_1 f(p_1 | w) dp_1 + p_{1j}^{\min} \int_{p_{1j}^{\min}}^{p_1^U} f(p_1 | w) dp_1 +$$

$$\sum_{m=1}^{j+1} c_{i_m^*} + t \quad 3-7-9$$

Rearranging (3-7-9) gives

$$\int_{p_1^L}^{p_{1j}^{\min}} (p_1 - p_{1j}^{\min}) f(p_1 | w) dp_1 + c_{i_{j+1}^*} \geq 0 \quad 3-7-10$$

Now consider the expected change in total expenditure from taking an additional observation on p_1 , given $(j+1)$ observations have already been taken

$$E[TE_{j+2}(\delta^*) | y_{j+1}] - TE_{j+1}(\delta^*) = \int_{p_1^L}^{p_{1,j+1}^{\min}} p_1 f(p_1 | w) dp_1 + p_{1,j+1}^{\min} \int_{p_{1,j+1}^{\min}}^{p_1^U} f(p_1 | w) dp_1$$

$$+ \sum_{m=1}^{j+2} c_{i_m^*} + t - p_{1,j+1}^{\min} - \sum_{m=1}^{j+1} c_{i_m^*} - t \quad 3-7-11$$

$$= \int_{p_1^L}^{p_{1,j+1}^{\min}} (p_1 - p_{1,j+1}^{\min}) f(p_1 | w) dp_1 + c_{i_{j+2}^*} \quad 3-7-12$$

Since $p_{1j}^{\min} \geq p_{1,j+1}^{\min}$ and since $c_{i_{j+2}^*} \geq c_{i_{j+1}^*}$ by (3-7-3),

$$E[TE_{j+2}(\delta^*)|y_{j+1}] - TE_{j+1}(\delta^*) > \int_{p_1^L}^{p_{1j}^{\min}} (p_1 - p_{1j}^{\min}) f(p_1|w) dp_1 + c_{i_{j+1}^*} \quad 3-7-13$$

$$\geq 0, \text{ by (3-7-10)} \quad 3-7-14$$

Therefore, if the searcher expects $TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*)|y_j]$, he also expects $TE_{j+1}(\delta^*) \leq E[TE_{j+2}(\delta^*)|y_{j+1}]$ and continuing in this manner shows he also expects

$$TE_{j+k}(\delta^*) \leq E[TE_{j+k+1}(\delta^*)|y_{j+k}] \text{ for all } k = 1, 2, \dots \quad 3-7-15$$

Comparing (3-7-15) to (3-7-7) shows

$$E[W_{j+1}(\xi^*, \delta^*)|y_j] = E[TE_{j+1}(\delta^*)|y_j] \quad 3-7-16$$

Therefore (3-7-1), (3-7-2) and (3-7-3) are sufficient conditions for the equivalence of (3-7-4) and (3-7-5).

Q.E.D.

Proposition 3-7-2:

If $w_{i_j} = w$ and $c_{i_j} = c > 0$ for all i_j , and if minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility, then the optimal sequential search rule ρ_s^* is myopic.

Proof:

$$\text{If } w_{i_j} = w \text{ for all } i_j \quad 3-7-17$$

$$\text{and if } c_{i_j} = c > 0 \text{ for all } i_j, \quad 3-7-18$$

then the optimal search path is given by Proposition 3-4-3. Therefore the sequence of transaction costs imposed by following the optimal search path of Proposition 3-4-3 is monotonic increasing

$$t_{i_j^*} \leq t_{i_{j+1}^*} \text{ for all } j = 1, \dots, j^*-1 \quad 3-7-19$$

The proof is similar to the proof of Proposition 3-7-1 in that conditions (3-7-17), (3-7-18) and (3-7-19) are shown to be sufficient for

$$E[W_{j+1}(\xi^*, \delta^*) | y_j] = E[TE_{j+1}(\delta^*) | y_j] \quad 3-7-20$$

$$\text{if } TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j]$$

$$\text{Suppose } TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \quad 3-7-21$$

Under conditions (3-7-17) and (3-7-18), (3-7-21) may be written as

$$\begin{aligned} (p_1+t)_j^{\min} + jc &\leq \int_{p_1^L}^{(p_1+t)_j^{\min-t_{i_{j+1}^*}}} (p_1+t_{i_{j+1}^*}) f(p_1|w) dp_1 \\ &+ (p_1+t)_j^{\min} \int_{(p_1+t)_j^{\min-t_{i_{j+1}^*}}}^{p_1^U} f(p_1|w) dp_1 + (j+1)c \end{aligned} \quad 3-7-22$$

Rearranging (3-7-22) gives

$$\int_{p_1^L}^{(p_1+t)_j^{\min-t_{i_{j+1}^*}}} \{ (p_1+t_{i_{j+1}^*}) - (p_1+t)_j^{\min} \} f(p_1|w) dp_1 + c \geq 0 \quad 3-7-23$$

The expected change in total expenditure from taking an additional observation on p_1 , given $(j+1)$ observations on p_1 have already been taken, is

$$\begin{aligned} E[TE_{j+2}(\delta^*) | y_{j+1}] - TE_{j+1}(\delta^*) &= \int_{p_1^L}^{(p_1+t)_{j+1}^{\min-t_{i_{j+2}^*}}} (p_1+t_{i_{j+2}^*}) f(p_1|w) dp_1 \\ &+ (p_1+t)_{j+1}^{\min} \int_{(p_1+t)_{j+1}^{\min-t_{i_{j+2}^*}}}^{p_1^U} f(p_1|w) dp_1 + (j+2)c - (p_1+t)_{j+1}^{\min} - (j+1)c \end{aligned} \quad 3-7-24$$

$$= \int_{p_1^L}^{(p_1+t)_{j+1}^{\min} - t_{i_{j+2}^*}} \{ (p_1+t_{i_{j+2}^*}) - (p_1+t)_{j+1}^{\min} \} f(p_1|w) dp_1 + c \quad 3-7-25$$

Since $(p_1+t)_j^{\min} \geq (p_1+t)_{j+1}^{\min}$ and since $t_{i_{j+1}^*} \leq t_{i_{j+2}^*}$ by (3-7-19),

$$(p_1+t)_{j+1}^{\min} - t_{i_{j+2}^*} \leq (p_1+t)_j^{\min} - t_{i_{j+1}^*} \quad 3-7-26$$

From (3-7-25) and (3-7-26),

$$E[TE_{j+2}(\delta^*)|y_{j+1}] - TE_{j+1}(\delta^*) \geq \int_{p_1^L}^{(p_1+t)_j^{\min} - t_{i_{j+1}^*}} \{ (p_1+t_{i_{j+1}^*}) - (p_1+t)_j^{\min} \} f(p_1|w) dp_1 + c \quad 3-7-27$$

$$\geq 0, \text{ by (3-7-23)} \quad 3-7-28$$

Therefore, if the searcher expects $TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*)|y_j]$, he also expects $TE_{j+1}(\delta^*) \leq E[TE_{j+2}(\delta^*)|y_{j+1}]$. Continuing in this manner shows he also expects

$$TE_{j+k}(\delta^*) \leq E[TE_{j+k+1}(\delta^*)|y_{j+k}] \text{ for all } k = 1, 2, \dots \quad 3-7-29$$

As in Proposition 3-7-1, (3-7-29) shows $E[W_{j+1}(\xi^*, \delta^*)|y_j] = E[TE_{j+1}(\delta^*)|y_j]$ so that conditions (3-7-17), (3-7-18), (3-7-19) are sufficient for the equivalence of the myopic and optimal sequential search rules.

Q.E.D.

Proposition 3-7-3:

If $t_{i_j} = t$ and $c_{i_j} = c > 0$ for all i_j , and if minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility, then the optimal sequential

search rule ρ_S^* is myopic.

Proof:

$$\text{If } t_{i_j} = t \text{ for all } i_j \quad 3-7-30$$

$$\text{and if } c_{i_j} = c > 0 \text{ for all } i_j, \quad 3-7-31$$

then the optimal search path is given by Proposition 3-5-5. Therefore the sequence of values of the conditioning parameter w for the optimal search path of Proposition 3-5-5 is either

$$w_{i_j^*} = w(j) \text{ for all } j = 1, \dots, j^* \text{ if } \frac{\partial F(p_1 | w)}{\partial w} \leq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-7-32$$

$$\text{or } w_{i_j^*} = \bar{w}(j) \text{ for all } j = 1, \dots, j^* \text{ if } \frac{\partial F(p_1 | w)}{\partial w} \geq 0 \text{ for all } p_1 \in [p_1^L, p_1^U] \quad 3-7-33$$

The proof is similar to the proofs of Propositions 3-7-1 and 3-7-2 in that conditions (3-7-30), (3-7-31) and either of (3-7-32) or (3-7-33) are shown to be sufficient for

$$E[W_{j+1}(\xi^*, \delta^*) | y_j] = E[TE_{j+1}(\delta^*) | y_j] \quad 3-7-34$$

$$\text{if } TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j]$$

$$\text{Suppose } TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \quad 3-7-35$$

Under conditions (3-7-30) and (3-7-31), (3-7-35) may be written as

$$p_{1j}^{\min} + t + jc \leq \int_{p_1^L}^{p_{1j}^{\min}} p_1 f(p_1 | w_{i_{j+1}^*}) dp_1 + p_{1j}^{\min} \int_{p_{1j}^{\min}}^{p_1^U} f(p_1 | w_{i_{j+1}^*}) dp_1 + t + (j+1)c \quad 3-7-36$$

Rearranging (3-7-36) gives

$$\int_{p_1^L}^{p_{1j}^{\min}} (p_1 - p_{1j}^{\min}) f(p_1 | w_{i_{j+1}^*}) dp_1 + c \geq 0 \quad 3-7-37$$

The expected change in total expenditure from taking an additional observation on p_1 , given $(j+1)$ observations on p_1 have already been taken, is

$$E[TE_{j+2}(\delta^*) | y_{j+1}] - TE_{j+1}(\delta^*) = \int_{p_1^L}^{p_{1,j+1}^{\min}} p_1 f(p_1 | w_{i_{j+2}}^*) dp_1$$

$$+ p_{1,j+1}^{\min} \int_{p_{1,j+1}^{\min}}^{p_1^U} f(p_1 | w_{i_{j+2}}^*) dp_1 + (j+2)c + t - p_{1,j+1}^{\min} - (j+1)c - t \quad 3-7-38$$

$$= \int_{p_1^L}^{p_{1,j+1}^{\min}} (p_1 - p_{1,j+1}^{\min}) f(p_1 | w_{i_{j+2}}^*) dp_1 + c \quad 3-7-39$$

Since $p_1 - p_{1,j+1}^{\min} < 0$ for all $p_1 \leq p_{1,j+1}^{\min}$

$$E[TE_{j+2}(\delta^*) | y_{j+1}] - TE_{j+1}(\delta^*) \geq \int_{p_1^L}^{p_{1,j+1}^{\min}} (p_1 - p_{1,j+1}^{\min}) f(p_1 | w_{i_{j+1}}^*) dp_1 + c \quad 3-7-40$$

for either (3-7-32) or (3-7-33). Since $p_{1j}^{\min} \geq p_{1,j+1}^{\min}$,

$$E[TE_{j+2}(\delta^*) | y_{j+1}] - TE_{j+1}(\delta^*) \geq \int_{p_1^L}^{p_{1j}^{\min}} (p_1 - p_{1j}^{\min}) f(p_1 | w_{i_{j+1}}^*) dp_1 + c \quad 3-7-41$$

≥ 0 , by (3-7-37)

Therefore, if the searcher expects $TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j]$, he also expects $TE_{j+1}(\delta^*) \leq E[TE_{j+2}(\delta^*) | y_{j+1}]$. Continuing in this manner shows he also expects $TE_{j+k}(\delta^*) \leq E[TE_{j+k+1}(\delta^*) | y_{j+k}]$ for all $k = 1, 2, \dots$ 3-7-42

As in Propositions 3-7-1 and 3-7-2, (3-7-42) shows

$E[W_{j+1}(\xi^*, \delta^*) | y_j] = E[TE_{j+1}(\delta^*) | y_j]$ so that conditions (3-7-30), (3-7-31) and either of conditions (3-7-32) or (3-7-33) are sufficient for the equivalence of the myopic and optimal sequential search rules.

Q.E.D.

The usefulness of Propositions 3-7-1, 3-7-2 and 3-7-3 in describing optimal search paths is restricted by their concentration upon only one directional influence on search at a time. The search path optimal when all three directional influences are present would not be characterised by any one of monotonic increasing sequences of marginal financial search costs alone, or transaction costs alone, or expected selling prices alone, but by some combination of all of these. The results of Propositions 3-7-1, 3-7-2 and 3-7-3 all have the common effect of ensuring the sequence of marginal changes in total expenditure on commodity 1 expected by the searcher is monotonic increasing. Indeed, it seems a reasonable conjecture that, as implied by the optimal sequential stopping rule ρ_s^* , the optimal search path will be characterised by a monotonic decreasing sequence of marginal expected utilities if marginal search costs are independent. Without the assumption of independence between marginal search costs, transaction costs or p.d.f.'s of sellers' prices, the optimal search path will remain as the solution to the complex dynamic programming problem referred to in Section 3-2.

The major implication of this section is, therefore, that myopic search behaviour will be rational (optimal) far more frequently, and under more general search conditions, than a casual glance at search problems would at first indicate.

A myopic expenditure minimisation search rule is used throughout the whole of the next chapter. Chapter 4 contains the derivations of the p.m.f. of search lengths and the p.d.f. of observed minimum prices resulting from searchers' use of such a rule. This simple rule is used to avoid making the analysis of Chapter 4 more complex than it already is. Some justification for the use of the rule can, however, be obtained by appealing to the content of this chapter. Accordingly, the search conditions assumed to exist in Chapter 4 are sufficient for the myopic sequential search rule to be optimal.

CHAPTER IV
PROBABILITY DISTRIBUTIONS OF SEARCH LENGTHS AND
TRANSACTION PRICES

SECTION 4-1: MODELS OF PRICE DISTRIBUTIONS ARISING FROM SEARCH

Stigler introduced search as an economic activity to examine causes of persistent price dispersion. To date, only six models, by Hey [15], Telser [53], Rothschild [42], Axell [1], Manning [28] and Ioannides [18], allow persistent price dispersion in a market and attempt to specify the manner in which search may affect the extent of price dispersion in a market. Each model has a quite different approach to the problem.

Telser [53] uses the device of many isolated sub-markets, each with many buyers and sellers and each of which clears perfectly at a price which may vary from sub-market to sub-market. Movement between sub-markets is costly and gives rise to persistent price dispersion between sub-markets.

Rothschild [42] uses the statistical theory of the two-armed bandit problem to establish a model in which identical competitive sellers learn about consumers' reservation prices and adopt optimal (maximising expected profits) pricing policies. These policies eventually make it optimal for a seller to offer a fixed price for all time. Rothschild shows that different experiences with customers' acceptances and refusals of their prices are likely to cause the individual sellers' optimal pricing policies to differ in the fixed prices eventually offered. Price dispersion thus persists for all time. The consumers' decision-making is left unspecified.

Manning [28] models a seller whose aim is to obtain the highest price possible for a single unit of a commodity. The seller knows the p.d.f. of buyers' reservation prices and the number of searchers who will contact him within a fixed period. Manning shows that, if all searchers appear identical to the seller, then it is optimal for the seller to quote to successive searchers from a particular monotonic decreasing sequence of prices. Which price offer, if any, is accepted by a searcher depends upon the order in which buyers with different reservation prices approach the seller. Consumer decision making is again left unspecified.

Axell's sequential search model [1] has been partly discussed in chapters 2 and 3. In the second half of his paper, Axell uses his model as the basis for a Monte Carlo study in which relative frequency plots are obtained for search lengths and observed minimum prices. Comparative statics results for the underlying probability distributions of search lengths and observed minimum prices are inferred from changes arising in these plots when changes are made in parameters such as financial search costs and variance of the p.d.f. of selling prices. The actual probability distributions are derived in Sections 4-3, 4-4 and 4-5 of this thesis. Comparative statics results are derived from these distributions and are compared to Axell's results. As Axell notes [1, p. 91], a major deficiency in his model is the common simplifying assumption of passivity on the part of sellers, who are restricted to maintaining the same p.d.f. of selling prices throughout the duration of consumers' searches.

Hey [15] provides a description of a market in which the combined actions of consumers carrying out Stigleresque searches leaves a

Pareto p.d.f. of selling prices unchanged from one period to the next. The model differs from those previously discussed in that it assumes Stigleresque search is rational and that searchers' demands for the commodity are functions of the smallest observed selling price.

Ioannides [18] has constructed an explicitly dynamic model in which neither buyer nor seller is passive and in which the p.d.f. of selling and reservation prices may vary continuously over time. An equilibrium is defined as a state in which the numbers of buyers and sellers and the probability distributions of reservation and asking prices are all stationary. The equilibrium "is characterised by price dispersion and unsatisfied supply and demand" [18, p. 261]. Ioannides is unable to describe how the market may attain such an equilibrium but is able to point out some sources of persistent price dispersion apparent from his analysis. First, search takes time. Second, market participants change and, as they change, maintain a stock of ignorance about the market. Third, as time passes, information becomes obsolete.

The model constructed in this chapter is closest in objective to Axell's model [1]. For simplicity, it is assumed that sellers are passive, so that the p.d.f. of selling prices of commodity 1 $f(p_1|w)$ remains unaltered for the duration of all consumers' searches. The conditions of search are assumed to be such that a myopic search rule which minimises ex ante expected total expenditure on commodity 1 is optimal. Within these restrictions, the probability distributions of search lengths and minimum observed prices are derived. The empirical results on search lengths obtained by Telser [52] and Gastwirth [11] were mentioned in Sections 1-2 and 2-1. These results

are compared to the theoretical results obtained in Section 4-3.

Sections 4-6 and 4-7 discuss the relationships of the probability distributions of search lengths and minimum observed prices to the probability distribution of transaction prices and market equilibrium.

SECTION 4-2: THE SEQUENCE OF RESERVATION PRICES FOR MYOPIC OPTIMAL SEQUENTIAL SEARCH

In this chapter relative simplicity is achieved, at the expense of generality, by making assumptions sufficient for the analysis to be carried out in monetary terms.

Assumption; Minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility, 4-2-1

Assumptions (1-4-28) and (3-5-1), that $w_{ij} = w$ and $t_{ij} = t$ for all i, j , are also invoked ie, the searcher cannot distinguish between sellers on the basis of their pricing behaviour and all sellers have the same transaction cost. Together, assumptions (4-2-1), (1-4-28) and (3-5-1) make it optimal for the searcher to follow a search path characterised by a monotonic increasing sequence of the smallest marginal financial search costs (see Proposition 3-3-2). Proposition 3-7-1 shows that (4-2-1), (1-4-28) and (3-5-1) are sufficient for the optimal sequential search rule to be myopic. Under the conditions of Proposition 2-7-1, all consumers will search and will terminate their searches by purchasing a unit of commodity 1 from the seller who offers them the lowest of the selling prices observed during search. Aggregate market demand for units of commodity 1 is therefore equal to the number of consumers and the p.d.f. of transaction prices will be the p.d.f. of

minimum observed prices.

Assumption: There are only a finite number of sellers with associated

$$\text{marginal financial search costs } c_{ij} < p_1^U - E[p_1]. \quad 4-2-2$$

This assumption is needed because of the linearity of the consumer's indirect utility function w.r.t. p_1^{\min} , assumed by (4-2-1). It was

shown in Section 2-1 that sequential search rules of the type employed by Rothschild, Axell etc., with a constant marginal financial cost

of search c , present the problem that there is no guarantee that search will be stopped before $x_1^* = 1$ becomes non-feasible. In fact,

if the marginal financial costs of search are all a constant

$c < p_1^U - E[p_1]$ then, for any finite number of observations, there is always a finite ex ante probability that a searcher will choose to take

an extra observation. The only conclusion that can be reached on the finiteness of search length is that search will be of infinite length

only with probability zero. Assumption (4-2-2) requires that there is

only a finite number of sellers for whom the associated marginal financial

costs of search are sufficiently small for it to be possible for the

searcher to expect to lower his total expenditure on commodity 1 by

contacting them. (4-2-2) thus ensures that any search will be of

finite length,

The myopic sequential stopping rule is $\xi^* = (\xi_0^*, \xi_1^*, \dots, \xi_j^*, \dots)$

where $\xi_0^*(y_0) = S_0^* \equiv 0$ and

$$\xi^*(y_j) = S_j^* = \begin{cases} 0, & TE_j(\delta^*) > E[TE_{j+1}(\delta^*) | y_j] \\ 1, & TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \end{cases}, \text{ for all } j \geq 1 \quad 4-2-3$$

The value of the searcher's j th reservation price C_j is the value of p_{1j}^{\min} for which he would be indifferent between stopping his search after j observations on p_1 and taking his $(j+1)$ th observation on p_1 . The searcher will be indifferent if and only if

$$TE_j(\delta^*) = E[TE_{j+1}(\delta^*) | y_j] \text{ ie,}$$

$$p_{1j}^{\min} + t + \sum_{i=1}^j c_i = p_{1j}^{\min} \int_{p_1^{\min}}^U f_g(p_1 | y_j) dp_1 + \int_{p_1^L}^{p_{1j}^{\min}} p_1 \cdot f_g(p_1 | y_j) dp_1 + t + \sum_{i=1}^{j+1} c_i$$

4-2-4

Rearranging (4-2-4) gives

$$\int_{p_1^L}^{p_{1j}^{\min}} (p_{1j}^{\min} - p_1) f_g(p_1 | y_j) dp_1 = c_{j+1}$$

4-2-5

as the condition for indifference between stopping and continuing search after j observations on p_1 . The L.H.S. of (4-2-5) is the lowering in the minimum price expected from taking observation p_1^{j+1} , given a present minimum price of p_{1j}^{\min} . The R.H.S. of (4-2-5) is the marginal financial cost of observation p_1^{j+1} . Therefore, since (4-2-5) is an equality, the change in total expenditure on commodity 1 expected from taking observation p_1^{j+1} is zero. The consumer will therefore be indifferent between stopping his search after j observations on p_1 and taking observation p_1^{j+1} .

The L.H.S. of (4-2-5) is bounded below by zero (when $p_{1j}^{\min} = p_1^L$) and bounded above by $p_1^U - E[p_1 | y_j]$ (when $p_{1j}^{\min} = p_1^U$). If $c_{j+1} < p_1^U - E[p_1 | y_j]$ then there is a value $C_j \in [p_1^L, p_1^U]$ of p_{1j}^{\min} such that the equality (4-2-5) is satisfied. Lemma 4-2-2 shows C_j

is unique for a given vector y_j and a given marginal financial cost c_{j+1} . If $c_{j+1} > p_1^U - E[p_1|y_j]$, then there is no value $C_j \in [p_1^L, p_1^U]$ satisfying the equality (4-2-5), because the marginal financial cost c_{j+1} exceeds the gain expected from taking observation p_1^{j+1} for any value of $p_{1j}^{\min} \in [p_1^L, p_1^U]$. Observation p_1^{j+1} will, therefore, not be taken. Proposition 4-2-6 expresses the optimal sequential stopping rule in terms of these values C_j in the following way: observation p_1^{j+1} is not taken if and only if $p_{1j}^{\min} \leq C_j$. If this is to be true for any value of p_{1j}^{\min} when $c_{j+1} > p_1^U - E[p_1|y_j]$, then the value of C_j must be p_1^U .

Definition:

The searcher's j th reservation price is $C_j = C_j(c_{j+1}, y_j)$ where

$$\begin{aligned}
 \text{(i)} \quad & \int_{p_1^L}^{C_j} (C_j - p_1) f_g(p_1|y_j) dp_1 = c_{j+1}, \\
 & \text{if } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1|y_j) dp_1 > c_{j+1} \\
 \text{(ii)} \quad & C_j = p_1^U, \text{ if } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1|y_j) dp_1 \leq c_{j+1}
 \end{aligned}
 \tag{4-2-6}$$

Since $p_{1j}^{\min} \in [p_1^L, p_1^U]$, it follows from (4-2-5) and (4-2-6) that $C_j \in [p_1^L, p_1^U]$ also. In Proposition 4-2-6 it is shown that the stopping rule ξ^* can be expressed entirely in terms of the sequence of reservation prices C_1, \dots, C_j, \dots , but first it is necessary to examine the properties of the reservation prices. Observations on p_1 are assumed by (1-4-10) to be independently distributed. Therefore (i) of (4-2-6) may be written as

$$\int_w \left\{ \int_{p_1^L}^{C_j} (C_j - p_1) f(p_1 | w) dp_1 \right\} g(w | y_j) dw = c_{j+1} \quad 4-2-7$$

where $g(w | y_j)$ is the searcher's posterior p.d.f. on w after j observations on p_1 . In Appendix 2, $g(w | y_j)$ is defined as

$$g(w | y_j) = \frac{f(p_1^1, \dots, p_1^j | w) g(w)}{\int_w f(p_1^1, \dots, p_1^j | w) g(w) dw} = \frac{f(y_j | w) g(w)}{f_g(y_j)} \quad 4-2-8$$

This is an application of Bayes' rule. Bayes' rule is the basis of all Bayesian processes of refining uncertain knowledge of parameters (w in this case) by using feedback from observations (p_1^1, \dots, p_1^j in this case) containing information about the true value of w .

Since observations on p_1 are independently distributed, (4-2-8) may be rewritten as

$$g(w | y_j) = \frac{\prod_{i=1}^j f(p_1^i | w) g(w)}{\int_w \prod_{i=1}^j f(p_1^i | w) g(w) dw} \quad 4-2-9$$

It should be apparent from a comparison of (4-2-6) and (4-2-9) that C_j is a function parameterised by the next marginal financial search cost c_{j+1} and past observations p_1^1, \dots, p_1^j ie.

$$C_j = C_j(c_{j+1}, p_1^1, \dots, p_1^j) \quad 4-2-10$$

A searcher's reservation price after j observations on p_1 is therefore dependent upon the values of these observations. The dependence is due to the searcher's Bayesian updating of his original prior $g(w)$. If all his observations have been high relative to values he ex ante considered relatively likely, he will think it increasingly likely that his next observation on p_1 will be relatively high also. High

values for previous observations on p_1 are therefore disincentives for further search and will be reflected in higher reservation prices. This is proved in Lemma 4-2-4 subject to the following assumption.

Assumption: $\frac{\partial F_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^i} \leq 0$ for all $p_1^i \in [p_1^L, p_1^U]$ and all
 $i = 1, \dots, j$ 4-2-11

This assumption expresses the following rational Bayesian actions.

Observing a high price will cause the searcher to revise his posterior p.d.f. on w , $g(w | p_1^1, \dots, p_1^j)$, so that his subjective likelihood of observing other high prices in the future is increased. The searcher behaves in this fashion because these actions eventually cause his posterior p.d.f. on w to collapse to a limiting degenerate p.d.f. centred on the correct value of w (see De Groot [9, p. 202]). This writer suspects, but has not proved, that assumption (4-2-11) is redundant in that it may be a necessary consequence of assuming either of (3-5-5), that $\partial F(p_1 | w) / \partial w \leq 0$ for all $p_1 \in [p_1^L, p_1^U]$, or (3-5-6), that $\partial F(p_1 | w) / \partial w \geq 0$ for all $p_1 \in [p_1^L, p_1^U]$.

The information content to the searcher of any observation on p_1 is the same regardless of when the observation is taken because the p.d.f. of selling prices is static. The manner in which a particular set of observed prices modifies the searcher's p.d.f. on w is therefore independent of the order in which these values are observed. As a result, the value C_j of the j th reservation price, corresponding to a particular value of c_{j+1} and a particular set of observed prices p_1^1, \dots, p_1^j , is also independent of the order in which these values were observed. This is proved in Lemma 4-2-3.

In models where the functional form and parameters of $f(p_1|w)$ are known exactly to searchers, searchers do not have a prior p.d.f. $g(w)$ (alternatively, $g(w)$ may be considered to exist but to be degenerate at the correct value of w). In these models, (4-2-7) simplifies to

$$\int_{p_1^L}^{C_j} (C_j - p_1) f(p_1|w) dp_1 = c_{j+1} \quad 4-2-12$$

and C_j is a function only of c_{j+1} . This case is examined in Section 4-5.

In the analysis that follows in Sections 4-2, 4-3 and 4-4, the reservation price will mostly be referred to only as C_j for the sake of notational simplicity, but the symbol C_j must be understood to be an abbreviation of the full expression given in (4-2-10).

C_j is dependent upon the magnitude of c_{j+1} , the next marginal financial search cost which will be incurred if the searcher makes an additional observation on p_1 . The greater the value of c_{j+1} , the greater is the disincentive for taking observation p_1^{j+1} , and this should be reflected in a higher reservation price. C_j is proved to be strictly monotonic increasing w.r.t. c_{j+1} in Lemma 4-2-1, a result analogous to Kohn and Shavell's result [20, p. 112] that "The switchpoint falls (*the reservation price rises*[†]) with an increase in next-period expected search costs".

Lemma 4-2-1:

$$\frac{\partial C_j}{\partial c_{j+1}} \left\{ \begin{array}{l} \geq 1 \\ = 0 \\ \text{is undefined} \end{array} \right\} \text{ as } - \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1|y_j) dp_1 \left\{ \begin{array}{l} > c_{j+1} \\ < c_{j+1} \\ = c_{j+1} \end{array} \right.$$

[†]My Italics

Proof:

$$(i) \quad \text{Suppose} \quad \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | y_j) dp_1 > c_{j+1} \quad 4-2-13$$

By (i) of (4-2-6), (4-2-13) implies there is a value $C_j < p_1^U$ of C_j such that

$$\int_{p_1^L}^{C_j} (C_j - p_1) f_g(p_1 | y_j) dp_1 = c_{j+1} \quad 4-2-14$$

Totally differentiating (4-2-14) w.r.t. c_{j+1} gives

$$\int_{p_1^L}^{C_j} f_g(p_1 | y_j) dp_1 \cdot \frac{\partial C_j}{\partial c_{j+1}} = dc_{j+1} \quad 4-2-15$$

Rearranging (4-2-15) gives

$$\frac{\partial C_j}{\partial c_{j+1}} = \left(\int_{p_1^L}^{C_j} f_g(p_1 | y_j) dp_1 \right)^{-1} \geq 1 \quad 4-2-16$$

$$(ii) \quad \text{Suppose} \quad \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | y_j) dp_1 < c_{j+1} \quad 4-2-17$$

By (ii) of (4-2-6), (4-2-17) implies $C_j = p_1^U$. Hence

$$\frac{\partial C_j}{\partial c_{j+1}} = \frac{\partial p_1^U}{\partial c_{j+1}} = 0 \quad 4-2-18$$

$$(iii) \quad \text{Suppose} \quad \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | y_j) dp_1 = c_{j+1} = \tilde{c} \quad 4-2-19$$

By (ii) of (4-2-6), (4-2-19) implies $C_j = p_1^U$. However (4-2-16) and (4-2-18) together imply

$$\lim_{c_{j+1} \rightarrow \tilde{c}^-} \frac{\partial C_j}{\partial c_{j+1}} \neq \lim_{c_{j+1} \rightarrow \tilde{c}^+} \frac{\partial C_j}{\partial c_{j+1}} \quad 4-2-20$$

which implies the non-smoothness of C_j at $c_{j+1} = \tilde{c}$. Hence $\frac{\partial C_j}{\partial c_{j+1}}$ is undefined if (4-2-19) is true.

Q.E.D.

Lemma 4-2-2:

C_j is unique for given $p_1^1, \dots, p_1^j, c_{j+1}$.

Proof:

$$\text{Suppose } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) \cdot f_g(p_1 | y_j) dp_1 \leq c_{j+1}$$

Then, from (4-2-6), $C_j = p_1^U$ and is unique.

$$\text{Suppose } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) \cdot f_g(p_1 | y_j) dp_1 > c_{j+1}$$

$$\text{Then } \int_{p_1^L}^{p_{1j}^{\min}} (p_{1j}^{\min} - p_1) \cdot f_g(p_1 | y_j) dp_1 = c_{j+1} \text{ if } p_{1j}^{\min} = C_j(c_{j+1}, y_j) = C_j$$

4-2-21

Consider

$$Q(z) = \int_{p_1^L}^z (z - p_1) \cdot f_g(p_1 | y_j) dp_1 \quad 4-2-22$$

$$\frac{dQ(z)}{dz} = \int_{p_1^L}^z f_g(p_1 | y_j) dp_1 > 0 \quad 4-2-23$$

provided $f_g(p_1 | y_j) \neq 0$ for all $p_1 \in [p_1^L, z]$. Recall from (2-2-14)

that $c_{j+1} > 0$. From (4-2-21) it is clear that $f_g(p_1 | y_j) \neq 0$

for all $p_1 \in [p_1^L, C_j]$. Therefore, from (4-2-23),

$$\int_{p_1^L}^z f_g(p_1 | y_j) dp_1 > 0 \text{ for all } C_j \leq z < p_1^U \quad 4-2-24$$

(4-2-24) proves the L.H.S. of (4-2-21) is strictly monotonic increasing in p_{1j}^{\min} for all $C_j \leq p_{1j}^{\min} < p_1^U$. Hence C_j is unique.

Q.E.D.

Lemma 4-2-3:

The value $C_j = C_j(c_{j+1}, p_1^1, \dots, p_1^j)$ is independent of the ordering of p_1^1, \dots, p_1^j .

Proof:

Let $\tilde{p}_1^1, \dots, \tilde{p}_1^j$ and $\hat{p}_1^1, \dots, \hat{p}_1^j$ be two orderings of the same set $\{p_1^1, \dots, p_1^j\}$ of observations on p_1 .

$$\tilde{C}_j = C_j(c_{j+1}, \tilde{p}_1^1, \dots, \tilde{p}_1^j) \text{ and } \hat{C}_j = C_j(c_{j+1}, \hat{p}_1^1, \dots, \hat{p}_1^j)$$

are both unique, by Lemma 4-2-2, and are such that

$$\int_{p_1^L}^{\tilde{C}_j} (\tilde{C}_j - p_1) \cdot f_g(p_1 | \tilde{p}_1^1, \dots, \tilde{p}_1^j) dp_1 = \int_{p_1^L}^{\hat{C}_j} (\hat{C}_j - p_1) \cdot f_g(p_1 | \hat{p}_1^1, \dots, \hat{p}_1^j) dp_1 = c_{j+1} \quad 4-2-25$$

$$g(w | \tilde{p}_1^1, \dots, \tilde{p}_1^j) = \frac{\prod_{i=1}^j f(\tilde{p}_1^i | w) g(w)}{\int_{\omega} \prod_{i=1}^j f(\tilde{p}_1^i | w) g(w) dw}, \text{ by (4-2-9)}$$

$$= \frac{\prod_{i=1}^j f(\hat{p}_1^i | w) g(w)}{\int_{\omega} \prod_{i=1}^j f(\hat{p}_1^i | w) g(w) dw}, \text{ since } \{\tilde{p}_1^1, \dots, \tilde{p}_1^j\} = \{\hat{p}_1^1, \dots, \hat{p}_1^j\} \quad 4-2-26$$

$$= g(w | \hat{p}_1^1, \dots, \hat{p}_1^j) \quad 4-2-27$$

$$\text{Therefore, } f_g(p_1 | \tilde{p}_1^1, \dots, \tilde{p}_1^j) = \int_{\omega} f(p_1 | w) g(w | \tilde{p}_1^1, \dots, \tilde{p}_1^j) dw$$

$$= \int_{\omega} f(p_1 | w) g(w | \hat{p}_1^1, \dots, \hat{p}_1^j) dw = f_g(p_1 | \hat{p}_1^1, \dots, \hat{p}_1^j) \quad 4-2-28$$

Comparing (4-2-25) and (4-2-28) shows $\tilde{C}_j = \hat{C}_j$. The value C_j is therefore independent of the ordering of p_1^1, \dots, p_1^j .

Q.E.D.

Lemma 4-2-4:

If $\frac{\partial F_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^m} < 0$ for all $p_1 \in [p_1^L, p_1^U]$,

then $\frac{\partial C_j}{\partial p_1^m} \begin{cases} > 0 \\ = 0 \\ \text{is undefined} \end{cases}$ as $\int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | y_j) dp_1 \begin{cases} > c_{j+1} \\ < c_{j+1} \\ = c_{j+1} \end{cases}$

for any $m = 1, \dots, j$,

Proof:

(i) If $\int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | y_j) dp_1 > c_{j+1}$ then, by (i) of (4-2-6),

there is a value $C_j < p_1^U$ of C_j such that

$$\int_{p_1^L}^{C_j} (C_j - p_1) f_g(p_1 | p_1^1, \dots, p_1^j) dp_1 = c_{j+1} > 0 \quad 4-2-29$$

Therefore,

$$\frac{\partial}{\partial p_1^m} \int_{p_1^L}^{C_j} (C_j - p_1) f_g(p_1 | p_1^1, \dots, p_1^j) dp_1 = 0 \quad 4-2-30$$

ie.

$$\int_{p_1^L}^{C_j} \frac{\partial C_j}{\partial p_1^m} f_g(p_1 | p_1^1, \dots, p_1^j) dp_1 + \int_{p_1^L}^{C_j} (C_j - p_1) \frac{\partial f_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^m} dp_1 = 0 \quad 4-2-31$$

Rearranging (4-2-31) gives

$$\frac{\partial C_j}{\partial p_1^m} = \frac{\int_{p_1^L}^{C_j} (p_1 - C_j) \frac{\partial f_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^m} dp_1}{\int_{p_1^L}^{C_j} f_g(p_1 | p_1^1, \dots, p_1^j) dp_1} \quad 4-2-32$$

First consider the denominator of the R.H.S. of (4-2-32). (4-2-29)

requires that $C_j = C_j(c_{j+1}, p_1^1, \dots, p_1^m, \dots, p_1^j) > p_1^L$ and that $f_g(p_1 | p_1^1, \dots, p_1^j) > 0$ for at least some $p_1 \in [p_1^L, C_j]$. Hence

$$\int_{p_1^L}^{C_j} f_g(p_1 | p_1^1, \dots, p_1^j) dp_1 > 0 \quad 4-2-33$$

Now consider the numerator of the R.H.S. of (4-2-32). Since

$$\frac{\partial}{\partial p_1} (p_1 - C_j(c_{j+1}, p_1^1, \dots, p_1^j)) = 1 > 0 \quad 4-2-34$$

and since $\frac{\partial F_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^m} < 0$ for all $p_1 \in [p_1^L, p_1^U]$,

we may appeal to the reasoning of Lemma 3-5-1 to establish that

$$\int_{p_1^L}^{C_j} (p_1 - C_j) \frac{\partial f_g(p_1 | p_1^1, \dots, p_1^j)}{\partial p_1^m} dp_1 > 0 \quad 4-2-35$$

Comparing (4-2-35) and (4-2-33) to (4-2-32) shows $\frac{\partial C_j}{\partial p_1^m} > 0$

when $\int_{p_1^L}^{p_1^U} (p_1 - p_1^U) f_g(p_1 | y_j) dp_1 > c_{j+1}$.

(ii) If $\int_{p_1^L}^{p_1^U} (p_1 - p_1^U) f_g(p_1 | y_j) dp_1 < c_{j+1}$ then, by (ii) of (4-2-6),

$$C_j = C_j(c_{j+1}, p_1^1, \dots, p_1^m, \dots, p_1^j) = p_1^U \quad 4-2-36$$

$$\text{Hence } \frac{\partial C_j}{\partial p_1^m} = \frac{\partial p_1^U}{\partial p_1^m} = 0$$

4-2-37

$$(iii) \text{ Suppose } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | p_1^1, \dots, \tilde{p}_1^m, \dots, p_1^j) dp_1 = c_{j+1}.$$

Then, by (ii) of (4-2-6), $C_j = C_j(c_{j+1}, p_1^1, \dots, \tilde{p}_1^m, \dots, p_1^j) = p_1^U$.

However, (4-2-35) and (4-2-37) together imply

$$\lim_{p_1^m \rightarrow \tilde{p}_1^m -} \frac{\partial C_j}{\partial p_1^m} \neq \lim_{p_1^m \rightarrow \tilde{p}_1^m +} \frac{\partial C_j}{\partial p_1^m} \quad 4-2-38$$

which implies the non-smoothness of C_j at $p_1^m = \tilde{p}_1^m$. Hence $\frac{\partial C_j}{\partial p_1^m}$ is

undefined for $p_1^m = \tilde{p}_1^m$.

Q.E.D.

The results of Lemmas 4-2-2, 4-2-3 and 4-2-4, that C_j is unique w.r.t. a set of observations p_1^1, \dots, p_1^j and a marginal cost c_{j+1} , is invariant w.r.t. the order of the values of p_1^1, \dots, p_1^j and that $\partial C_j / \partial p_1^m > 0$, do not offer any information about an upper bound on the sensitivity of C_j w.r.t. a change in p_1^m . In general, it is not possible to specify such an upper bound because the sensitivity of C_j w.r.t. p_1^m depends crucially upon the form of the p.d.f. of selling prices and upon the searcher's prior beliefs about values of w . To demonstrate this dependence, consider the following example. Suppose only three selling prices are possible, \$1, \$49.99 and \$50.00. The searcher's prior on w is such that he believes that either the respective probabilities of these prices are 0.99, 0.01, 0.00 or 0.00, 0.00, 1.00. If the searcher's first observation is $p_1^1 = \$50.00$, he will regard this as confirmation of the second set of probabilities. However, a small change in the value of p_1^1 from \$50.00 to \$49.99 would

support the first set of probabilities. Because the second set of probabilities offer no gain from search, the searcher would set a high reservation price so as to stop searching. However, the first set of probabilities mean it is very likely that a price lower than \$49.99 will be observed and the searcher will set a low reservation price so as to continue search. A small change in p_1^1 may therefore cause a large change in C_1 , the value of the first reservation price C_1 . Nevertheless, some limits upon the sensitivity of C_j w.r.t. p_1^m do exist. Lemma 4-2-5 shows that $\partial C_j / \partial p_1^m < 1$ for some values of $p_1^m \in [p_1^L, p_1^U]$. Note that this does not preclude the possibility of $\partial C_j / \partial p_1^m \geq 1$ for some values of $p_1^m \in [p_1^L, p_1^U]$, as in the example above.

Lemma 4-2-5:

$$\frac{\partial C_j}{\partial p_1^m} < 1, \quad 1 \leq m \leq j, \quad \text{for some values of } p_1^m \in [p_1^L, p_1^U].$$

Proof:

Suppose $\frac{\partial C_j}{\partial p_1^m} \geq 1$ for all $p_1^m \in [p_1^L, p_1^U]$. Then

$$\int_{p_1^L}^{p_1^U} \frac{\partial C_j}{\partial p_1^m} dp_1^m \geq \int_{p_1^L}^{p_1^U} 1 \cdot dp_1^m \quad 4-2-39$$

$$\begin{aligned} \text{ie. } C_j(c_{j+1}, p_1^1, \dots, p_1^{m-1}, p_1^U, p_1^{m+1}, \dots, p_1^j) - C_j(c_{j+1}, p_1^1, \dots, p_1^{m-1}, p_1^L, p_1^{m+1}, \dots, p_1^j) \\ \geq p_1^U - p_1^L \end{aligned} \quad 4-2-40$$

But, $C_j \leq p_1^U$, by (4-2-6), and, since $c_{j+1} > 0$, $C_j > p_1^L$, by (4-2-21).

Hence

$$|C_j(c_{j+1}, \tilde{y}_j) - C_j(c_{j+1}, \hat{y}_j)| < p_1^U - p_1^L \quad 4-2-41$$

for any vectors of observations \tilde{y}_j and \hat{y}_j . (4-2-40) and (4-2-41) are contradictory, so the result is established.

Q.E.D.

The following proposition proves that the optimal myopic stopping rule ξ^* may be expressed in terms of the sequence of reservation prices C_j . The reader must not confuse this with what has been called the "reservation price property". The reservation price property applies to the narrow class of search problems in which

(i) the p.d.f. of selling prices $f(p_1|w)$ is completely known to the searcher, and

(ii) the marginal cost of all observations is the same. In these search problems, the optimal sequential stopping rule is to continue searching until an observed price is either equal to or less than a single particular price called the "reservation price". Problems of this type are discussed more fully in Section 4-5. In problems of the type considered in Sections other than 4-5, Bayesian learning about an incompletely known p.d.f. $f(p_1|w)$ ensures that no one such reservation price exists. As expressed by (4-2-10), the reservation prices are functions of observed selling prices. While the decision to halt or extend search by one more observation is made by comparison of observed prices to a reservation price, the same reservation price is not used for each such decision. Kohn and Shavell [20, p. 93] point out, in a sequential search framework more general than the one in use here, that switchpoints (reservation prices) always characterise an optimal sequential stopping rule.

Rothschild [43, p. 701] provides a simple numerical example demonstrating this distinction.

Proposition 4-2-6:

The myopic sequential stopping rule is $\xi^* = (\xi_0^*, \xi_1^*, \dots, \xi_j^*)$

where

$$\begin{aligned} \xi_0^*(y_0) &= S_0^* \equiv 0 \text{ and} \\ \xi_j^*(y_j) &= S_j^* = \begin{cases} 0, & p_{1j}^{\min} > C_j(c_{j+1}, p_1^1, \dots, p_1^j) \\ 1, & p_{1j}^{\min} \leq C_j(c_{j+1}, p_1^1, \dots, p_1^j) \end{cases} \end{aligned}$$

for all $j = 1, 2, \dots$.

Proof:

(4-2-3) gives the j th component, $j \geq 1$, of the myopic sequential stopping rule as

$$\xi_j^*(y_j) = S_j^* = \begin{cases} 0, & TE_j(\delta^*) > E[TE_{j+1}(\delta^*) | y_j] \\ 1, & TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j] \end{cases} \quad 4-2-42$$

Suppose $TE_j(\delta^*) \leq E[TE_{j+1}(\delta^*) | y_j]$. Then, from (4-2-4) and (4-2-5),

$$\int_{p_1^L}^{p_{1j}^{\min}} (p_{1j}^{\min} - p_1) \cdot f_g(p_1 | y_j) dp_1 \leq c_{j+1} \quad 4-2-43$$

By (4-2-23) and (4-2-24), the L.H.S. of (4-2-43) is a function monotonic increasing in p_{1j}^{\min} and strictly monotonic increasing for values of p_{1j}^{\min} in the neighbourhood of C_j . Lemma 4-2-2 proves C_j is unique. Therefore (4-2-43) is true if and only if $p_{1j}^{\min} \leq C_j$. Hence $S_j^* = 1$ if and only if $p_{1j}^{\min} \leq C_j$. Similarly $S_j^* = 0$ if and only if $p_{1j}^{\min} > C_j$.

Q.E.D.

The manner in which (4-2-6) is solved for the form of the reservation price functions C_j is not discussed because it will depend upon the specific functional forms of $f(p_1 | w)$ and $g(w)$.

The actual value C_j of any reservation price C_j will be determinate only after the j th observation on p_1 . For a given search path, the likelihoods of different values $C_1, C_2, \dots, C_j, \dots$ for the reservation prices $C_1, C_2, \dots, C_j, \dots$ are related to the likelihoods of different sets of values for observed prices, and thereby to the p.d.f. of selling prices. The p.d.f. of selling prices is therefore related to the probabilities that, after j observations on p_1 , the least observed price will, or will not, exceed the j th reservation price, a function of the observed prices. It is this relationship which allows the derivation of the p.m.f. of search lengths and the p.d.f. of minimum observed prices in terms of the p.d.f. of selling prices and the marginal financial search costs of the chosen search path.

The next section presents the derivation of the ex ante p.m.f. of search lengths and examines its properties.

SECTION 4-3: THE EX ANTE PROBABILITY MASS FUNCTION OF SEQUENTIAL SEARCH LENGTHS

The reader is reminded of the point, made in Section 3-3, that the term "search length" is unambiguous only in the context of sequential search. In frameworks other than that of sequential search, it is necessary to distinguish between the number of periods T_i in which search is carried out, and the number of observations taken. In this chapter, as in Chapters 2 and 3, the term search length is retained because the sequential framework used causes the number of periods searched, and the number of observations taken, to be the same. Using the term search length will also facilitate comparison

of this work with the work of previous authors.

The p.m.f. of search lengths, ex ante or otherwise, is trivial in the case of Stigleresque search (where search length can only be interpreted as the number of observations taken). If the Stigleresque searcher decides ex ante to make j observations on p_1 , the p.m.f. of search lengths has mass one for length j and mass zero for any other length. The difficulty in deriving a p.m.f. of search lengths for a sequential searcher is that the search length depends upon the values of the selling prices observed during search. It is even more complicated if the searcher has incomplete knowledge of the p.d.f. of selling prices, since his sequence of decisions on continuing or halting search is then made with differing amounts of information about the true p.d.f. of selling prices he faces. The case examined here is that of a searcher with incomplete knowledge of the p.d.f. of selling prices and an optimal sequential search rule which is myopic and which minimises ex ante expected total expenditure on commodity 1. The p.m.f. is of interest not merely for its own sake, but because it crucially influences the p.d.f. of observed minimum prices. For instance, a p.m.f. of search lengths skewed towards higher search lengths means that, on average, more observations are taken on p_1 , so that the p.d.f. of minimum observed prices will be skewed towards lower prices.

The starting point for the derivation of the p.m.f. of search lengths is the form of the optimal stopping rule given in Lemma 4-2-6. Recall from (1-4-21) that search is halted at the end of period T_j if and only if $\psi_j^* = 1$. The probability of an optimal sequential search of length j is, therefore, $\Pr(\psi_j^* = 1|w)$. Three comments must be made

about this probability at the outset. First, it is the probability that the searcher will observe prices p_1^1, \dots, p_1^j such that he will choose to stop searching without taking a $(j+1)$ th observation on p_1 . The searcher's own assessment of this p.m.f. will coincide with the actual p.m.f. if and only if the true value of w is known to him. Second, the probabilities are conditional upon the true value of w because the sequence of decisions to halt or continue search depends upon observed prices, the probabilities of which are conditional upon w . Third, the searcher does not know the true value of w , although he is able to refine his knowledge of this value as search proceeds. How refined his knowledge of w becomes depends, in part, on the accuracy of his initial knowledge of w i.e. upon his initial prior p.d.f. on w , $g(w)$. The probabilities $\Pr(\psi_j^* = 1|w)$ are, therefore, also conditional upon $g(w)$. (4-21) states that $\psi_j^* = 1$ if and only if $S_0^* = S_1^* = \dots = S_{j-1}^* = 0$ and $S_j^* = 1$. Therefore,

$$\Pr(\psi_j^* = 1|w) = \Pr(S_0^* = S_1^* = \dots = S_{j-1}^* = 0, S_j^* = 1|w) \quad 4-3-1$$

for all $j \geq 0$. Note that

$$\Pr(\psi_0^* = 1|w) = \Pr(S_0^* = 1|w) = 0 \quad 4-3-2$$

since, under the conditions of Proposition 2-7-1, $S_0^* \equiv 0$.

Therefore,

$$\begin{aligned} \Pr(\psi_j^* = 1|w) &= \Pr(S_1^* = \dots = S_{j-1}^* = 0, S_j^* = 1|w) \cdot \Pr(S_0^* = 0|w) \\ &= \Pr(S_1^* = \dots = S_{j-1}^* = 0, S_j^* = 1|w) \end{aligned} \quad 4-3-3$$

(4-3-3) can be expressed in terms of minimum observed prices and reservation prices by noting, from Proposition 4-2-6, that for all $i = 1, \dots, j-1$, $S_i^* = 0$ if and only if $p_{1i}^{\min} > C_i$, and $S_j^* = 1$ if and only if $p_{1j}^{\min} \leq C_j$. The probability that search is of length j is the probability that the values of the j observed prices are such that

all these events occur ie.

$$\Pr(\psi_j^* = 1|w) = \Pr(p_{11}^{\min} > C_1, \dots, p_{1,j-1}^{\min} > C_{j-1}, p_{1j}^{\min} \leq C_j | w) \quad 4-3-4$$

for all $j \geq 1$. It is emphasised that the probabilities (4-3-4) are ex ante. In particular, they must be distinguished from the searcher's assessment, after taking j observation on p_1 , of the probability of his taking a $(j+1)$ th observation on p_1 . The latter probability is either zero or unity since the searcher can observe with certainty whether or not $p_{1j}^{\min} > C_j$. Certain sets of j observation on p_1 will cause this probability to be zero. All others will cause it to be unity. The linear combination of these values 0 or 1, weighted by the ex ante likelihoods of the sets of j observations that give rise to them, is just the searcher's ex ante expectation of the value of ψ_j^* . The equivalence of this expectation and the ex ante probability (4-3-4) is shown by (4-3-5).

$$E[\psi_j^* | w] = 0 \cdot \Pr(\psi_j^* = 0 | w) + 1 \cdot \Pr(\psi_j^* = 1 | w) = \Pr(\psi_j^* = 1 | w) \quad 4-3-5$$

The searcher's decision-making after j observations on p_1 utilises his j th posterior p.d.f. on w , $g(w|y_j)$, a function of p_1^1, \dots, p_1^j . In expressing (4-3-4) in terms of values of price observations p_1^1, \dots, p_1^j , therefore, account must be taken of the effects different values for p_1^1, \dots, p_1^j will have upon the searcher's revision of his initial prior p.d.f. on w , $g(w)$. The revision of the prior is reflected in the searcher's revision of the values of his reservation prices C_1, \dots, C_j , so (4-3-4) must take account of the different values possible for the reservation prices. To see how the likelihoods of possible reservation price values are related to the p.d.f. of selling prices $f(p_1|w)$, consider when, after one observation on p_1 , the searcher decides that

$$p_{11}^{\min} = p_1^1 > C_1$$

$$\text{ie. } p_1^1 > C_1(c_2, p_1^1) \quad 4-3-6$$

If $C_1 = p_1^U$, then $p_1^1 > C_1$ is impossible and search will halt after the first observation on p_1 . If $C_1 < p_1^U$, there is some non-zero probability that (4-3-6) is true. Lemma 4-2-4 shows that, if $C_1 < p_1^U$, C_1 is strictly monotonic increasing w.r.t. p_1^1 . Lemma 4-3-1 shows that if, in addition, $\partial C_1 / \partial p_1^1 < 1$ for all $p_1^1 \in [p_1^L, p_1^U]$, then there is a unique value of p_1^1 , $p_1^* \in (p_1^L, p_1^U]$, such that

$$p_1^* = C_1(c_2, p_1^*) \quad 4-3-7$$

and that $p_1^1 > C_1(c_2, p_1^1)$ if and only if $p_1^1 > p_1^*$. The value of p_1^* summarises the effects the possible values of p_1^1 have upon the value of C_1 and upon the likelihood of the searcher deciding to take more than a single observation on p_1 . Lemma 4-3-1 and Proposition 4-3-2 extend this argument to reservation prices subsequent to C_1 , and show that the ex ante p.m.f. of search lengths may be expressed as the probabilities of joint events determined by the values of a sequence p_1^*, \dots, p_j^* defined as follows.

Definition:

The set of solutions to the simultaneous equations

$$p_1^i = C_i(c_{i+1}, p_1^1, \dots, p_1^i), \text{ for all } i = 1, \dots, j \quad 4-3-8$$

is $p_1^i = p_1^*$ where $p_1^* \in (p_1^L, p_1^U]$ for all $i = 1, \dots, j$

Uniqueness of the set of solutions p_1^*, \dots, p_j^* can be guaranteed by the following assumption.

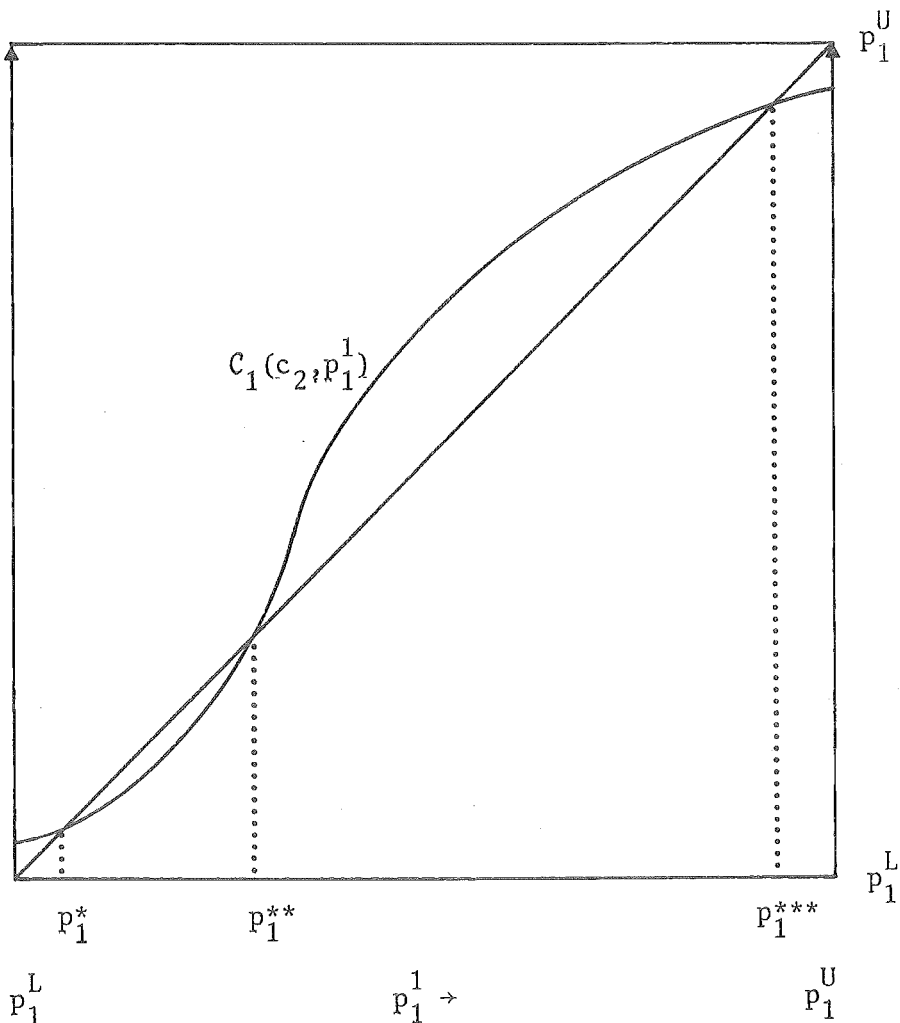
Assumption: $\partial C_i / \partial p_1^m < 1$ for all $p_1^m \in [p_1^L, p_1^U]$ and for all

$$m = 1, \dots, i; i = 1, 2, \dots \quad 4-3-9$$

The result of Lemma 4-2-5, that $\partial C_i / \partial p_1^m < 1$ for some values of $p_1^m \in [p_1^L, p_1^U]$, is not sufficient to guarantee uniqueness. Figure

4-3-1 illustrates a case where three solutions $p_1^*, p_1^{**}, p_1^{***}$ exist for the equation $p_1^1 = C_1(c_2, p_1^1)$. Non-uniqueness may arise without (4-3-9) because the result of Lemma 4-2-5 does not rule out the possibility that $\partial C_1 / \partial p_1^m > 1$ for some $p_1^m \in [p_1^L, p_1^U]$. Non-uniqueness among the p_1^* greatly complicates the derivation of the ex ante p.m.f. of search lengths, so assumption (4-3-9) is invoked throughout the rest of this chapter. This is equivalent to assuming small changes in values of observed prices cannot cause revisions in the searcher's beliefs about values of w that are so great as to cause large changes in any of the reservation prices,

Figure 4-3-1



Lemma 4-3-1:

- (i) p_1^*, \dots, p_j^* exist
- (ii) (4-3-9) is sufficient for p_1^*, \dots, p_j^* to be unique.

Proof:

An illustration of the proof is contained in Figure 4-3-2. The proof of existence is presented first for p_1^* and is achieved by applying Brouwer's theorem. Note

(a) the interval $[p_1^L, p_1^U]$ is a non-empty, compact and convex subset of R^1 4-3-10

(b) $C_1(p_1^1) \leq p_1^U$, for all $p_1^1 \in [p_1^L, p_1^U]$ by (4-2-6), and $C_1(p_1^1) > p_1^L$ for all $p_1^1 \in [p_1^L, p_1^U]$ by (4-2-21). Hence

$$C_1 : [p_1^L, p_1^U] \rightarrow (p_1^L, p_1^U] \subset [p_1^L, p_1^U] \quad 4-3-11$$

(4-3-10) and (4-3-11) show two of Brouwer's theorem's three sufficiency conditions are satisfied. The third sufficiency condition is that C_1 is continuous w.r.t. p_1^1 for all $p_1^1 \in [p_1^L, p_1^U]$. The proof of the continuity of C_1 w.r.t. p_1^1 is in two parts.

(i) Suppose c_2 is sufficiently large for $C_1(c_2, p_1^1) = p_1^U$ for all $p_1^1 \in [p_1^L, p_1^U]$. Then it is a trivial exercise to prove C_1 continuous w.r.t. p_1^1 for all $p_1^1 \in [p_1^L, p_1^U]$. 4-3-12

(ii) Suppose c_2 is such that

$$\int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | \hat{p}_1) dp_1 = c_2 \quad 4-3-13$$

for some $p_1^1 = \hat{p}_1 \in [p_1^L, p_1^U]$. Then, by (4-2-6),

$$C_1(c_2, \hat{p}_1) = p_1^U \quad 4-3-14$$

$$\text{Also, } \frac{\partial}{\partial \hat{p}_1} \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | \hat{p}_1) dp_1 = \int_{p_1^L}^{p_1^U} (p_1^U - p_1) \frac{\partial f_g(p_1 | \hat{p}_1)}{\partial \hat{p}_1} dp_1 < 0 \quad 4-3-15$$

by Lemma 3-5-1 and assumption (4-2-11) since $\frac{\partial}{\partial p_1}(p_1^U - p_1) = -1 < 0$.

Therefore,

$$\int_{p_1^L}^{p_1^U} (p_1^U - p_1) f_g(p_1 | p_1^1) dp_1 > c_2 \text{ as } p_1^1 < \hat{p}_1 \quad 4-3-16$$

(4-3-16) and (4-2-6) imply

$$C_1(c_2, p_1^1) = p_1^U \text{ for all } p_1^1 > \hat{p}_1 \quad 4-3-17$$

and

$$C_1(c_2, p_1^1) < p_1^U \text{ for all } p_1^1 < \hat{p}_1 \quad 4-3-18$$

The reasoning in (i) above establishes that C_1 is continuous w.r.t. p_1^1 for all $p_1^1 \in (\hat{p}_1, p_1^U]$. Lemma 4-2-4 establishes that $\frac{\partial C_1}{\partial p_1^1}$ exists for

$C_1(c_2, p_1^1) < p_1^U$. Therefore, by (4-3-18), C_1 is continuous w.r.t. p_1^1 for all $p_1^1 \in [p_1^L, \hat{p}_1)$. It remains to show that C_1 is continuous w.r.t. p_1^1 for $p_1^1 = \hat{p}_1$. By (4-3-17),

$$\lim_{p_1^1 \rightarrow \hat{p}_1^+} C_1(c_2, p_1^1) = \lim_{p_1^1 \rightarrow \hat{p}_1^+} p_1^U = p_1^U \quad 4-3-19$$

C_1 is bounded above by p_1^U and below by p_1^L so, by the Lebesgue Dominated Convergence theorem,

$$\begin{aligned} & \lim_{p_1^1 \rightarrow \hat{p}_1} \int_{p_1^L}^{C_1(c_2, p_1^1)} (C_1(c_2, p_1^1) - p_1) f_g(p_1 | p_1^1) dp_1 \\ &= \int_{p_1^L}^{\lim_{p_1^1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1)} \left(\lim_{p_1^1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) - p_1 \right) f_g(p_1 | \hat{p}_1) dp_1 \end{aligned} \quad 4-3-20$$

Suppose

$$\int_{p_1^L}^{\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1)} (\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) - p_1) f_g(p_1 | \hat{p}_1) dp_1 < c_2 \quad 4-3-21$$

$$\text{Then, by (4-3-13), } \lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) < p_1^U \quad 4-3-22$$

but, by (4-3-16), (4-3-22) implies there exists

$$C_1(c_2, p_1^1) < p_1^U \text{ for some } p_1^1 > \hat{p}_1, \quad 4-3-23$$

(4-3-23) contradicts (4-3-17) so (4-3-21) is false.

Suppose

$$\int_{p_1^L}^{\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1)} (\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) - p_1) f_g(p_1 | \hat{p}_1) dp_1 > c_2 \quad 4-3-24$$

$$\text{Then, by (4-3-13), } \lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) > p_1^U \quad 4-3-25$$

but this contradicts (4-2-6) so (4-3-24) is false. Hence

$$\int_{p_1^L}^{\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1)} (\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) - p_1) f_g(p_1 | \hat{p}_1) dp_1 = c_2 \quad 4-3-26$$

Comparing (4-3-13) and (4-3-26) shows

$$\lim_{p_1 \rightarrow \hat{p}_1} C_1(c_2, p_1^1) = p_1^U \quad 4-3-27$$

Combining (4-3-14), (4-3-19) and (4-3-27) proves C_1 is continuous w.r.t.

p_1^1 for $p_1^1 = \hat{p}_1$. Hence C_1 is continuous w.r.t. p_1^1 for all

$p_1^1 \in [p_1^L, p_1^U]$. This proves the third sufficiency condition of Brouwer's

theorem so there exists a fixed point $p_1^* \in [p_1^L, p_1^U]$ such that

$$p_1^* = C_1(c_2, p_1^*) \quad 4-3-28$$

The fixed points p_i^* are used in the derivation of the ex ante probability distributions of search lengths and minimum observed prices. It is, therefore, important to establish that the p_i^* can be evaluated prior to search. It is clear from (4-3-7) that the value of p_1^* depends on the value of c_2 alone ie.

$$p_1^* = \phi_1(c_2) \quad 4-3-31$$

Similarly, p_2^* depends only on c_2 and c_3 ie.

$$p_2^* = C_2(c_3, p_1^*, p_2^*) = C_2(c_3, \phi_1(c_2), p_2^*) \quad 4-3-32$$

$$\text{so } p_2^* = \phi_2(c_2, c_3) \quad 4-3-33$$

$$\text{In general, } p_i^* = \phi_i(c_2, \dots, c_{i+1}) \quad 4-3-34$$

The searcher has complete knowledge of all costs so it is clear from (4-3-34) that he has all the information required to evaluate the p_i^* .

(4-3-4) expresses the ex ante probability of a search length of j in terms of the reservation prices C_1, \dots, C_j . The probability may be rewritten as

$$\Pr(\psi_j^* = 1 | w) = \Pr(p_{1j}^{\min} \leq C_j | w, p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j-1). \quad 4-3-35$$

$$\Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j-1 | w)$$

The following Proposition expresses (4-3-35) in terms of the solutions p_i^* and completes the derivation of the ex ante p.m.f. of search lengths. Expressing the p.m.f. in terms of the p_i^* captures the searcher's assessments of the effects of different samples of observed prices upon his re-evaluations of his prior p.d.f. on w , and of his sequence of reservation prices. It is noted above that, since the p_i^* are functions of only c_2, \dots, c_{i+1} , the values of the p_i^* can be determined ex ante. Remember, however, that once an observation on

p_1 has been taken, the searcher will recalculate a new sequence of solutions p_2^*, p_3^*, \dots in an assessment of a p.m.f. of search lengths of two or more, conditioned by the value of the first price observation. A similar updating of the p.m.f. of search lengths will take place after every price observation until search is halted.

Proposition 4-3-2:

If minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility and if $w_{i_j} = w$ and $t_{i_j} = t$ for all i_j , then the ex ante probability mass function of optimal sequential search lengths is

$$\Pr(\psi_j^* = 1 | w) = \begin{cases} F(p_1^* | w), & \text{for } j = 1 \\ \prod_{i=1}^{j-1} (1 - F(\max\{p_i^*, \dots, p_{j-1}^*\} | w)) \cdot \prod_{i=1}^j (1 - F(\max\{p_i^*, \dots, p_j^*\} | w)), & \text{for } j \geq 2. \end{cases}$$

Proof:

First consider $j = 1$. By (4-3-9), $p_1^1 \leq C_1(c_2, p_1^1)$ if and only if $p_1^1 \leq p_1^*$. Therefore

$$\begin{aligned} \Pr(\psi_1^* = 1 | w) &= \Pr(p_{11}^{\min} \leq C_1 | w) = \Pr(p_1^1 \leq C_1(c_2, p_1^1) | w) \\ &= \Pr(p_1^1 \leq p_1^* | w) \\ &= F(p_1^* | w) \end{aligned} \quad 4-3-36$$

Now consider $j \geq 2$. From (4-3-35),

$$\Pr(\psi_j^* = 1 | w) = (1 - \Pr(p_{1j}^{\min} > C_j | w, p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j-1)).$$

$$\Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j-1 | w) \quad 4-3-37$$

$$= \Pr(p_{1i}^{\min} > C_i \text{ for all } i=1, \dots, j-1 | w) - \Pr(p_{1i}^{\min} > C_i \text{ for all } i=1, \dots, j | w)$$

4-3-38

$$\Pr(p_{1i}^{\min} > C_i \text{ for all } i=1, \dots, j | w)$$

$$= \Pr(p_1^1 > C_1(c_2, p_1^1), \min(p_1^1, p_1^2) > C_2(c_3, p_1^1, p_1^2), \dots, \min(p_1^1, \dots, p_1^j) > C_j(c_{j+1}, p_1^1, \dots, p_1^j) | w)$$

4-3-39

By (4-3-9), $p_1^1 > C_1(c_2, p_1^1)$ if and only if $p_1^1 > p_1^* = C_1(c_2, p_1^*)$. Again by (4-3-9), $p_1^1 > p_1^*$ if and only if

$$C_2(c_3, p_1^1, p_1^2) > C_2(c_3, p_1^*, p_1^2), \dots, C_j(c_{j+1}, p_1^1, p_1^2, \dots, p_1^j) > C_j(c_{j+1}, p_1^*, p_1^2, \dots, p_1^j)$$

4-3-40

(4-3-39) may therefore be rewritten as

$$\begin{aligned} & \Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j | w) \\ &= \Pr(p_1^1 > p_1^*, \min(p_1^1, p_1^2) > C_2(c_3, p_1^*, p_1^2), \dots, \min(p_1^1, \dots, p_1^j) > C_j(c_{j+1}, p_1^*, p_1^2, \dots, p_1^j) | w) \end{aligned}$$

4-3-41

If $\min(p_1^1, p_1^2) > C_2(c_3, p_1^*, p_1^2)$, then both $p_1^1, p_1^2 > C_2(c_3, p_1^*, p_1^2)$.

But, by (4-3-9), $p_1^2 > C_2(c_3, p_1^*, p_1^2)$ if and only if

$p_1^2 > p_2^* = C_2(c_3, p_1^*, p_2^*)$. Therefore $\min(p_1^1, p_1^2) > C_2(c_3, p_1^*, p_1^2)$ if and only if both $p_1^1, p_1^2 > p_2^*$. (4-3-41) may therefore be rewritten as

$$\begin{aligned} & \Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j | w) \\ &= \Pr(p_1^1 > p_1^*, \min(p_1^1, p_1^2) > p_2^*, \min(p_1^1, p_1^2, p_1^3) > C_3(c_4, p_1^*, p_2^*, p_1^3), \dots, \\ & \dots, \min(p_1^1, \dots, p_1^j) > C_j(c_{j+1}, p_1^*, p_2^*, p_1^3, \dots, p_1^j) | w) \end{aligned}$$

4-3-42

Continuing the above reasoning shows

$$\begin{aligned} & \Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j | w) = \Pr(p_1^1 > p_1^*, \min(p_1^1, p_1^2) > p_2^*, \dots, \\ & \min(p_1^1, \dots, p_1^j) > p_j^* | w) \end{aligned}$$

4-3-43

The event described in (4-3-43) is true if and only if p_1^1 is greater than all of $p_1^*, p_2^*, \dots, p_j^*$ and if and only if p_1^2 is greater than all of p_2^*, \dots, p_j^* and so on. Hence,

$$\Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j | w) \\ = \Pr(p_1^1 > \max\{p_1^*, \dots, p_j^*\}, p_1^2 > \max\{p_2^*, \dots, p_j^*\}, \dots, p_1^j > p_j^* | w) \quad 4-3-44$$

$$= \prod_{i=1}^j (1 - F(\max\{p_1^*, \dots, p_j^*\} | w)) \quad 4-3-45$$

since all observations are assumed to be independently distributed by (1-4-10). Similarly,

$$\Pr(p_{1i}^{\min} > C_i \text{ for all } i = 1, \dots, j-1 | w) = \prod_{i=1}^{j-1} (1 - F(\max\{p_1^*, \dots, p_{j-1}^*\} | w)) \quad 4-3-46$$

Substituting (4-3-46) and (4-3-45) into (4-3-38) gives, for $j \geq 2$,

$$\Pr(\psi_j^* = 1 | w) = \prod_{i=1}^{j-1} (1 - F(\max\{p_1^*, \dots, p_{j-1}^*\} | w)) - \prod_{i=1}^j (1 - F(\max\{p_1^*, \dots, p_j^*\} | w)) \quad 4-3-47$$

Q.E.D.

Before continuing, it is worthwhile noting that eventually a $p_j^* = p_1^U$. In Section 4-2 it was noted that assumption (4-2-2) meant that there were only a finite number of sellers for whom a searcher would set a reservation price below p_1^U . For all other sellers, $C_j = p_1^U$ regardless of the vector of observations taken up to this stage of the search. For these sellers therefore,

$$p_j^* = C_j(c_{j+1}, p_1^*, \dots, p_{j-1}^*, p_j^*) = p_1^U \quad 4-3-48$$

Remark 4-3-3:

It is an easy matter to show that the p.m.f. derived in Proposition 4-3-2 satisfies the fundamental requirement of a proper p.m.f., that its components sum to unity.

Proof:

$$\sum_{j=1}^{\infty} \Pr(\psi_j^* = 1 | w) = \lim_{J \rightarrow \infty} \sum_{j=1}^J \Pr(\psi_j^* = 1 | w) \quad 4-3-49$$

From Proposition 4-3-2,

$$\begin{aligned} \sum_{j=1}^J \Pr(\psi_j^* = 1 | w) &= F(p_1^* | w) + (1 - F(p_1^* | w)) \\ &\quad \cdot \prod_{i=1}^2 (1 - F(\max\{p_i^*, p_2^*\} | w)) \\ &\quad + \prod_{i=1}^2 (1 - F(\max\{p_i^*, p_2^*\} | w)) - \dots \\ &\quad \dots \cdot \prod_{i=1}^J (1 - F(\max\{p_i^*, \dots, p_J^*\} | w)) \end{aligned} \quad 4-3-50$$

$$= 1 - \prod_{i=1}^J (1 - F(\max\{p_i^*, \dots, p_J^*\} | w)) \quad 4-3-51$$

(4-3-48) shows that eventually a $p_j^* = p_1^U$. Suppose $p_{J'}^* = p_1^U$.

From (4-3-51) and (4-3-49),

$$\sum_{j=1}^{\infty} \Pr(\psi_j^* = 1 | w) = 1 - \lim_{J \rightarrow \infty} \prod_{i=1}^J (1 - F(\max\{p_i^*, \dots, p_J^*\} | w)) \quad 4-3-52$$

$$= 1 - \lim_{J \rightarrow \infty} \prod_{i=1}^{J'} (1 - F(\max\{p_i^*, \dots, p_{J'}^*, \dots, p_J^*\} | w)). \quad 4-3-53$$

$$\prod_{i=J'+1}^J (1 - F(\max\{p_i^*, \dots, p_J^*\} | w))$$

$$= 1 - \lim_{J \rightarrow \infty} (1 - F(p_1^U | w))^{J'} \prod_{i=J'+1}^J (1 - F(\max\{p_i^*, \dots, p_J^*\} | w)) \quad 4-3-54$$

$$= 1$$

Q.E.D.

This completes the derivation of the ex ante p.m.f. of search lengths. The p.m.f. depends upon the magnitudes of the marginal financial search costs and the sequence in which they will be incurred. Consequently, the p.m.f. is dependent upon the searcher's choice of

search path. At the same time, the choice of search path is made by comparing the expected total expenditures on commodity 1 associated with them. These expected expenditures are calculated with the use of the ex ante p.m.f. corresponding to the search path under consideration. Earlier in this section it was pointed out that, as search proceeds and observations on p_1 accumulate, the p.m.f. of search lengths will be reassessed by the searcher. The reassessment of the p.m.f. of search lengths is, therefore, necessarily accompanied by a reassessment of the optimal search path.

The following lemmas and Propositions examine some of the properties of the ex ante p.m.f. Lemma 4-3-4 proves p_j^* is a strictly monotonic increasing function of each of c_2, \dots, c_{j+1} . This result is useful in Propositions 4-3-5 and 4-3-6, which provide some comparative statics results for the p.m.f. of search lengths.

The first result of Proposition 4-3-5 is that the p.m.f. of search lengths is unaffected by a change in the first marginal financial search cost c_1 . As explained in Section 3-3, the reason for this is that, under the conditions of Proposition 2-7-1, a searcher will always choose to search. c_1 is therefore incurred whatever the length of search undertaken and can never influence the length of search for which total expenditure on commodity 1 is minimised. Since expenditure minimisation is the criterion by which the advantages of different search lengths are judged, and since a change in c_1 does not alter any of these advantages, the p.m.f. of search lengths is unaffected by a change in c_1 . The remaining results of Proposition 4-3-5 are that an increase in the $(j+1)$ th marginal financial search cost c_{j+1} , $j \geq 1$, may increase the probability that search is of length j , may decrease

the probability that search is longer than j observations, and does not affect any of the probabilities of search being of lengths 1 to $(j-1)$ observations on p_1 . The explanation is as follows. An increase in c_{j+1} is an extra disincentive to take the $(j+1)$ th observation on p_1 . The probability that a searcher will choose not to take this observation is therefore increased. Because the optimal search rule is myopic, the decision on whether or not to take a $(j+1)$ th observation on p_1 is made only *after* j observations on p_1 have been taken. Consequently, only the probability that search is of length j is increased, thereby causing a corresponding decrease in the probability that search is of any length of more than j observations on p_1 .

The ex ante expected length of search is

$$E[j|w] = \sum_{j=1}^{\infty} j \cdot \Pr(\psi_j^* = 1|w) \quad 4-3-55$$

An immediate implication of the changes caused in the p.m.f. of search lengths by an increase in c_{j+1} is that an increase in c_{j+1} , $j \geq 1$, may cause a decrease in the ex ante expected search length. This is the result of Proposition 4-3-6.

Lemma 4-3-4:

$$\frac{dp_j^*}{dc_k} \begin{cases} = 0, & \text{for all } k > j+1, \text{ or } k=1 \\ \geq 0, & \text{for all } k \leq j+1, k \neq 1 \end{cases}$$

Proof:

By (4-3-34), p_j^* is a unique function of c_2, \dots, c_{j+1} .

$$\text{Hence } \frac{dp_j^*}{dc_1} = 0 \quad 4-3-56$$

$$\text{and } \frac{dp_j^*}{dc_k} = 0, \text{ for all } k > j+1 \quad 4-3-57$$

Let $k = 2$. Consider $p_1^* = \phi_1(c_2) = C_1(c_2, p_1^*)$.

$$\frac{dp_1^*}{dc_2} = \frac{\partial C_1}{\partial c_2} + \frac{\partial C_1}{\partial p_1^*} \frac{dp_1^*}{dc_2} \quad 4-3-58$$

$$\frac{dp_1^*}{dc_2} = \frac{\partial C_1}{\partial c_2} / (1 - \frac{\partial C_1}{\partial p_1^*}) \quad 4-3-59$$

$$\text{By Lemma 4-2-1, } \frac{\partial C_1}{\partial c_2} \geq 0 \quad 4-3-60$$

$$\text{By (4-3-9) and Lemma 4-2-4, } 1 \geq (1 - \frac{\partial C_1}{\partial p_1^*}) > 0 \quad 4-3-61$$

Combining (4-3-59), (4-3-60) and (4-3-61) shows

$$\frac{dp_1^*}{dc_2} \geq 0 \quad 4-3-62$$

Now consider $p_2^* = \phi_2(c_2, c_3) = C_2(c_3, p_1^*, p_2^*)$

$$\frac{dp_2^*}{dc_2} = \frac{\partial C_2}{\partial p_1^*} \frac{dp_1^*}{dc_2} + \frac{\partial C_2}{\partial p_2^*} \frac{dp_2^*}{dc_2} \quad 4-3-63$$

$$\frac{dp_2^*}{dc_2} = \frac{\partial C_2}{\partial p_1^*} \cdot \frac{dp_1^*}{dc_2} / (1 - \frac{\partial C_2}{\partial p_2^*}) \quad 4-3-64$$

$$\text{By (4-3-9) and Lemma 4-2-4, } \frac{\partial C_2}{\partial p_1^*} \geq 0, 1 \geq (1 - \frac{\partial C_2}{\partial p_2^*}) > 0 \quad 4-3-65$$

Combining (4-3-65), (4-3-64) and (4-3-62) shows

$$\frac{dp_2^*}{dc_2} \geq 0 \quad 4-3-66$$

In general, for any $i = 1, \dots, j$,

$$\frac{dp_i^*}{dc_2} = \sum_{j=1}^{i-1} \frac{\partial C_i}{\partial p_j^*} \cdot \frac{dp_j^*}{dc_2} / (1 - \frac{\partial C_i}{\partial p_i^*}) \geq 0 \quad 4-3-67$$

since $\frac{\partial C_i}{\partial p_j^*} \geq 0, 1 \geq (1 - \frac{\partial C_i}{\partial p_i^*}) > 0$ and $\frac{dp_1^*}{dc_2}, \dots, \frac{dp_{i-1}^*}{dc_2} \geq 0$.

$$\text{Similarly, } \frac{dp_i^*}{dc_k} = \sum_{j=k-1}^{i-1} \frac{\partial C_i}{\partial p_j^*} \cdot \frac{dp_j^*}{dc_k} / (1 - \frac{\partial C_i}{\partial p_i^*}) \quad 4-3-68$$

for any $2 \leq k \leq j+1$, and, by beginning with $\frac{dp_{k-1}^*}{dc_k}$, it may be shown that $\frac{dp_i^*}{dc_k} \geq 0$ for all $i = k-1, \dots, j$.

Q.E.D.

Proposition 4-3-5:

- (i) $\frac{d}{dc_1}(\Pr(\psi_j^* = 1|w)) = 0$, for all $j \geq 1$.
- (ii) $\frac{d}{dc_k}(\Pr(\psi_j^* = 1|w)) = 0$, for all $k > j+1$, all $j \geq 1$.
- (iii) $\frac{d}{dc_{j+1}}(\Pr(\psi_j^* = 1|w)) \geq 0$, for all $j \geq 1$.
- (iv) $\frac{d}{dc_{j+1}}(\sum_{k=j+1}^{\infty} \Pr(\psi_k^* = 1|w)) \leq 0$, for all $j \geq 1$.

Proof:

(i) and (ii): From Proposition 4-3-2;

$$\text{For } j = 1: \frac{d}{dc_k}(\Pr(\psi_1^* = 1|w)) = \frac{d}{dc_k}(F(p_1^*|w)) = 0 \quad 4-3-69$$

if $k = 1$ or $k > 2$ since $\frac{dp_1^*}{dc_k} = 0$ for $k = 1$ or $k > 2$ by Lemma 4-3-4.

The case $k = 2$ is dealt with in (iii) below.

$$\text{For } j \geq 2: \frac{d}{dc_k}(\Pr(\psi_j^* = 1|w)) = \frac{d}{dc_k}(\prod_{i=1}^{j-1} (1-F(\max\{p_i^*, \dots, p_{j-1}^*\}|w)))$$

$$= \prod_{i=1}^j (1-F(\max\{p_i^*, \dots, p_j^*\}|w))$$

$$= 0$$

4-3-70

if $k > j+1$ since all of $\frac{dp_1^*}{dc_k} = \dots = \frac{dp_j^*}{dc_k} = 0$ if $k > j+1$ by Lemma 4-3-4.

(iii) From Proposition 4-3-2;

$$\frac{d}{dc_2}(\Pr(\psi_1^* = 1|w)) = \frac{d}{dc_2}(F(p_1^*|w)) = f(p_1^*|w) \frac{dp_1^*}{dc_2} \geq 0 \quad 4-3-71$$

since $\frac{dp_1^*}{dc_2} \geq 0$ by Lemma 4-3-4.

$$\text{For } j \geq 2: \frac{d}{dc_{j+1}}(\Pr(\psi_j^* = 1|w)) = \frac{d}{dc_{j+1}} \left(\prod_{i=1}^{j-1} (1-F(\max\{p_i^*, \dots, p_{j-1}^*\}|w)) \right)$$

$$- \prod_{i=1}^j (1-F(\max\{p_i^*, \dots, p_j^*\}|w))]$$

$$= \sum_{n=1}^j f(\max\{p_n^*, \dots, p_j^*\}|w) \frac{d}{dc_{j+1}}(\max\{p_n^*, \dots, p_j^*\}) \prod_{\substack{i=1 \\ i \neq n}}^j (1-F(\max\{p_i^*, \dots, p_j^*\}|w))$$

4-3-72

$$\text{since } \frac{dp_1^*}{dc_{j+1}} = \dots = \frac{dp_{j-1}^*}{dc_{j+1}} = 0 \text{ by Lemma 4-3-4. From (4-3-72),}$$

$$\frac{d}{dc_{j+1}}(\Pr(\psi_j^* = 1|w)) \geq 0$$

4-3-73

$$\text{since } \frac{dp_j^*}{dc_{j+1}} \geq 0 \text{ by Lemma 4-3-4,}$$

(iv)

$$\frac{d}{dc_{j+1}} \left(\sum_{k=j+1}^{\infty} \Pr(\psi_k^* = 1|w) \right) = \frac{d}{dc_{j+1}} (1 - \Pr(\psi_j^* = 1|w) - \sum_{k=1}^{j-1} \Pr(\psi_k^* = 1|w))$$

4-3-74

$$= - \frac{d}{dc_{j+1}}(\Pr(\psi_j^* = 1|w)) - \sum_{k=1}^{j-1} \frac{d}{dc_{j+1}}(\Pr(\psi_k^* = 1|w))$$

4-3-75

$$= - \frac{d}{dc_{j+1}}(\Pr(\psi_j^* = 1|w)), \text{ by (ii) above}$$

4-3-76

$$\leq 0, \text{ by (iii) above.}$$

4-3-77

Q.E.D.

Proposition 4-3-6:

$$\begin{aligned} \frac{d}{dc_k}(E[j|w]) &\leq 0, \text{ for all } k \geq 2 \\ &= 0, \text{ for } k = 1 \end{aligned}$$

Proof:

From (4-3-55),

$$\frac{d}{dc_k}(E[j|w]) = \sum_{j=1}^{\infty} j \cdot \frac{d}{dc_k}(\Pr(\psi_j^* = 1|w)) \quad 4-3-78$$

(i) of Proposition 4-3-5 gives $\frac{d}{dc_1}(\Pr(\psi_j^* = 1|w)) = 0$ for all $j \geq 1$. Hence,

$$\frac{d}{dc_1}(E[j|w]) = 0 \quad 4-3-79$$

Consider any value of $k \geq 2$. From (4-3-55) and Proposition 4-3-2

$$\begin{aligned} \frac{d}{dc_k}(E[j|w]) &= 1 \cdot f(p_1^*|w) \cdot \frac{dp_1^*}{dc_k} \\ &+ \sum_{j=2}^{\infty} j \left[\sum_{n=1}^j f(\max\{p_n^*, \dots, p_j^*\}|w) \cdot \frac{d}{dc_k}(\max\{p_n^*, \dots, p_j^*\}) \cdot \prod_{\substack{i=1 \\ i \neq n}}^j (1-F(\max\{p_i^*, \dots, p_j^*\}|w)) \right. \\ &- \left. \sum_{n=1}^{j-1} f(\max\{p_n^*, \dots, p_{j-1}^*\}|w) \cdot \frac{d}{dc_k}(\max\{p_n^*, \dots, p_{j-1}^*\}) \cdot \prod_{\substack{i=1 \\ i \neq n}}^{j-1} (1-F(\max\{p_i^*, \dots, p_{j-1}^*\}|w)) \right] \end{aligned} \quad 4-3-80$$

$$\begin{aligned} &= - \sum_{j=2}^{\infty} \sum_{n=1}^{j-1} f(\max\{p_n^*, \dots, p_{j-1}^*\}|w) \cdot \frac{d}{dc_k}(\max\{p_n^*, \dots, p_{j-1}^*\}) \\ &\quad \prod_{\substack{i=1 \\ i \neq n}}^{j-1} (1-F(\max\{p_i^*, \dots, p_{j-1}^*\}|w)) \end{aligned} \quad 4-3-81$$

$$\begin{aligned} &= - \sum_{j=k-1}^{\infty} \sum_{n=1}^{j-1} f(\max\{p_n^*, \dots, p_{j-1}^*\}|w) \cdot \frac{d}{dc_k}(\max\{p_n^*, \dots, p_{j-1}^*\}) \\ &\quad \prod_{\substack{i=1 \\ i \neq n}}^{j-1} (1-F(\max\{p_i^*, \dots, p_{j-1}^*\}|w)) \end{aligned} \quad 4-3-82$$

since $\frac{dp_n^*}{dc_k} = 0$ for all $n < k-1$ by Lemma 4-3-4. Also,

$\frac{dp_n^*}{dc_k} \geq 0$ for all $n \geq k-1$, by Lemma 4-3-4. Combining this with (4-3-82)

shows $\frac{d}{dc_k}(E[j|w]) \leq 0$.

Q.E.D.

These results are compared to some of the empirical results obtained by Axell [1], Gastwirth [11] and Telser [52] and to the results of Proposition 3-3-3 and Proposition 3-3-4. Telser [52] concluded that an increase in the marginal financial cost of search c reduced the gain expected from search. Gastwirth [11] concluded that increasing c reduced the ex ante expected length of search. Axell [1] concluded that increasing c increased both the mean and variance of the ex ante p.d.f. of minimum prices found during search. These results are similar to the theoretical results presented here, but there is one essential difference. The comparative statics results presented in this chapter refer to a change in only one of the marginal financial search costs at a time, Axell's, Gastwirth's and Telser's assumption of a constant marginal financial search cost allows them to consider only changes of the same amount in all such costs at once. Their results are, therefore, efforts to measure the combined effects of simultaneous changes of the same amount in all marginal financial search costs. The quantitative results obtained are similar to the results predicted by the analysis of this section. The two sets of results are still directly comparable though, because each of the marginal results derived here is shown to push in the same direction.

Consequently, the joint effect of combining all the individual marginal results will have the same qualitative behaviour as each marginal result.

Both Proposition 3-3-4 and Proposition 4-3-6 show the ex ante expected search length is non-increasing w.r.t. any of the marginal financial search costs.

The results of Propositions 3-3-3 and 4-3-5, while not at variance with each other, are not completely the same. They are similar in that both show that the ex ante p.m.f. of search lengths is unaffected by a change in c_1 , that an increase in c_{j+1} increases the probability of a search length of j observations on p_1 , and that an increase in c_{j+1} decreases the probability of a search length of greater than j observations. However, the two sets of results do not completely coincide in two respects. Firstly, Proposition 3-3-3 shows that, for $k \geq 2$,

$$\partial \Pr(\psi_j^* = 1 | w) / \partial c_k \geq 0, \text{ for all } j = 1, \dots, k-2 \quad 4-3-83$$

while Proposition 4-3-5 shows

$$\partial \Pr(\psi_j^* = 1 | w) / \partial c_k = 0, \text{ for all } j = 1, \dots, k-2 \quad 4-3-84$$

The proof of Proposition 4-3-5 recognises that, under the assumed conditions of search, the optimal search rule is myopic. A change in c_k does not affect the searcher's decision-making until he has taken $(k-1)$ observations on p_1 . This is what is expressed by (4-3-84).

The proof of Proposition 3-3-3, however, does not utilise myopia.

As a result, all of the marginal financial search costs, except c_1 , influence the searcher's decisions at all stages of the search. A

change in c_k can therefore alter the searcher's decision on whether or not to take an additional observation on p_1 at some stage prior to his

having taken $(k-1)$ observations. An increase in c_k is an increase in a search penalty which may be incurred at a later stage of the search and so is a disincentive for additional search. This is what is expressed by (4-3-83).

Secondly, Proposition 3-3-3 shows that, for $k \geq 2$,

$$\partial \Pr(\psi_j^* = 1 | w) / \partial c_k \leq 0, \text{ for all } j \geq k \quad 4-3-85$$

while Proposition 4-3-5 shows

$$\partial \left(\sum_{j=k}^{\infty} \Pr(\psi_j^* = 1 | w) \right) / \partial c_k \leq 0 \quad 4-3-86$$

This writer has not succeeded in showing the stronger result of (4-3-85) in the context of Proposition 4-3-5.

The next section contains the derivation of the p.d.f. of minimum observed prices and an examination of its properties.

SECTION 4-4: THE EX ANTE PROBABILITY DENSITY FUNCTION OF MINIMUM OBSERVED PRICES

The search literature shows considerable interest in the distribution of searchers' minimum observed prices because it is related in some way to the distribution of prices at which market transactions occur (see Rothschild [41] for a survey of such articles). The effects of changes in the conditions of search may be traced through the distributions of search lengths and minimum observed prices to the distribution of transaction prices and, from there, to the effects on the distribution of selling prices in the next period. An understanding of the processes by which these changes are affected will assist economists' understanding of their theorised adjustment processes for a sequence of market equilibria. In this section the p.d.f. of minimum observed prices is derived and its properties examined. In the case

treated here, the distribution of minimum observed prices and the distribution of transaction prices coincide because the conditions of search cause all searchers to purchase one, and only one, unit of commodity 1. In other, more realistic, situations, the relationship of the distribution of transaction prices to the distribution of minimum observed prices will be less simple. For instance, the searcher's freedom to choose not to buy and the presence of transaction costs will cause the transaction prices distribution to be a version of the distribution of minimum observed prices that is attenuated at the high price end,

In the Stigleresque search case, the number j of observations taken on p_1 is fixed prior to search and never revised. Appendix 4 shows that the resulting p.d.f. of minimum observed prices is the p.d.f. of the minimum of a sample of size j drawn from a population with p.d.f. $f(p_1|w)$ ie.

$$h(p_1^{\min}|w) = j(1-F(p_1^{\min}|w))^{j-1}f(p_1^{\min}|w) \quad 4-4-1$$

The sequential search case has two major complications. First, and well recognised, is that the length of search is a random variable which is a function of observed prices. Second, and hitherto completely ignored, is that, unlike the Stigleresque expression (4-4-1), the sequential p.d.f. of minimum observed prices must take into account the order in which different selling prices are observed. For example, suppose there are only two selling prices in the market, \$10 and \$20. A searcher whose first observation $p_1^1 = \$20$ may well decide to take a second observation. Suppose the value of this observation is $p_1^2 = \$10$. The minimum of this sample of size 2 is \$10. However, a sample of size 2 with a minimum of \$10 is irrational if it was

collected in the order $p_1^1 = \$10$, $p_1^2 = \$20$. The first observation of \$10 would have caused the searcher to halt his search immediately since the expected net gain from taking another observation is negative.

Proposition 4-4-1 presents the derivation of the cumulative p.d.f. of minimum observed prices, $H(p_1^{\min}|w)$.

Proposition 4-4-1:

If minimisation of ex ante expected total expenditure on commodity 1 is a valid proxy for maximisation of ex ante expected utility and if $w_{i_j} = w$ and $t_{i_j} = t$ for all i_j , then the cumulative density function of minimum observed prices $p_1^{\min} \in [p_1^L, p_1^U]$ is

$$H(p_1^{\min}|w) = 1 - (F(\max\{p_1^{\min}, p_1^*\}|w) - F(p_1^{\min}|w)) \\ - \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\}|w) - F(p_1^{\min}|w)) \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\}|w))$$

Proof:

$H(p_1^{\min}|w)$ is the probability that the smallest observed price does not exceed p_1^{\min} whenever search is halted. In order to derive the above c.d.f., consider instead $(1 - H(p_1^{\min}|w))$, the probability that the smallest observed price does exceed p_1^{\min} when search is halted. The searcher halts his search after his first observation if and only if $p_1^1 \leq p_1^*$. Similarly, the searcher halts his search after two observations if and only if $p_1^1 > p_1^*$ and $p_1^2 \leq p_2^*$. In general, search is of length j if and only if the searcher's first j observations are such that $p_1^1 > p_1^*, \dots, p_1^{j-1} > p_{j-1}^*, p_1^j \leq p_j^*$. Therefore,

$$1 - H(p_1^{\min}|w) = \Pr(p_1^{\min} > p_1^{\min} \text{ and } p_1^1 \leq p_1^*)$$

$$\text{or } p_{12}^{\min} > p_1^{\min} \text{ and } p_1^1 > p_1^*, p_1^2 \leq p_2^* \quad 4-4-2$$

.....

$$\text{or } p_{1j}^{\min} > p_1^{\min} \text{ and } p_1^1 > p_1^*, \dots, p_1^{j-1} > p_{j-1}^* \text{ and } p_1^j \leq p_j^*$$

$$\text{or } \dots | w)$$

(4-4-2) is the probability of the union of a sequence of mutually exclusive events. Lemma 2-5-4 proved all such sequences to be of finite length. Since the events are mutually exclusive,

$$\begin{aligned} 1-H(p_1^{\min} | w) &= \Pr(p_1^{\min} > p_1^{\min} \text{ and } p_1^1 \leq p_1^* | w) \\ &+ \sum_{j=2}^{\infty} \Pr(p_{1j}^{\min} > p_1^{\min} \text{ and } p_1^1 > p_1^*, \dots, p_1^{j-1} > p_{j-1}^* \text{ and } p_1^j \leq p_j^* | w) \quad 4-4-3 \end{aligned}$$

$$\begin{aligned} &= \Pr(p_1^1 > p_1^{\min} \text{ and } p_1^1 \leq p_1^* | w) \\ &+ \sum_{j=2}^{\infty} \Pr(p_1^1 > p_1^{\min}, \dots, p_1^j > p_1^{\min}, p_1^1 > p_1^*, \dots, p_1^{j-1} > p_{j-1}^*, p_1^j \leq p_j^* | w) \quad 4-4-4 \end{aligned}$$

$$\begin{aligned} &= \Pr(p_1^{\min} < p_1^1 \leq p_1^* | w) \\ &+ \sum_{j=2}^{\infty} \Pr(p_1^1 > \max\{p_1^{\min}, p_1^*\}, \dots, p_1^{j-1} > \max\{p_1^{\min}, p_{j-1}^*\}, p_1^{\min} < p_1^j \leq p_j^* | w) \quad 4-4-5 \end{aligned}$$

Assumption (1-4-10) is that all observations on p_1 are independently distributed.

Hence,

$$1-H(p_1^{\min} | w) = \Pr(p_1^{\min} < p_1^1 \leq p_1^* | w) + \sum_{j=2}^{\infty} \prod_{i=1}^{j-1} \Pr(p_1^i > \max\{p_1^{\min}, p_i^*\} | w).$$

$$\Pr(p_1^{\min} < p_1^j \leq p_j^* | w) \quad 4-4-6$$

$$\begin{aligned}
&= F(\max\{p_1^{\min}, p_1^*\} | w) - F(p_1^{\min} | w) + \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \\
&\quad \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \quad 4-4-7
\end{aligned}$$

Rearranging (4-4-7) gives the desired expression.

Q.E.D.

The following remarks verify some of the properties essential for $H(p_1^{\min} | w)$ to be a proper c.d.f.

Remark 4-4-2:

$$H(p_1^L | w) = 0$$

Proof:

From Proposition 4-4-1, if $p_1^{\min} = p_1^L$,

$$H(p_1^L | w) = 1 - F(p_1^* | w) - \sum_{j=2}^{\infty} \prod_{i=1}^{j-1} (1 - F(p_i^* | w)) \cdot F(p_j^* | w) \quad 4-4-8$$

$$= 1 - F(p_1^* | w) - (1 - F(p_1^* | w)) \cdot F(p_2^* | w) - (1 - F(p_1^* | w)) (1 - F(p_2^* | w)) \cdot F(p_3^* | w) \quad 4-4-9$$

...

$$= [1 - F(p_1^* | w)], [1 - F(p_2^* | w) - (1 - F(p_2^* | w))], F(p_3^* | w), \dots] \quad 4-4-10$$

$$= [1 - F(p_1^* | w)][1 - F(p_2^* | w)][1 - F(p_3^* | w)][\dots]$$

$$= \prod_{j=1}^{\infty} [1 - F(p_j^* | w)] \quad 4-4-11$$

(4-3-48) noted that assumption (4-2-2) was sufficient to guarantee that only a finite number of p_j^* could have values less than p_1^U . Therefore, eventually a $p_j^* = p_1^U$. From (4-4-11) it is easily seen that $H(p_1^L | w) = 0$ if any one term

$$[1 - F(p_j^* | w)] = [1 - F(p_1^U | w)] = 0$$

Q.E.D.

Remark 4-4-3:

$$H(p_1^U | w) = 1,$$

Proof:

From Proposition 4-4-1, for any $p_1^{\min} \in [p_1^L, p_1^U]$,

$$\begin{aligned} H(p_1^{\min} | w) &= 1 - (F(\max\{p_1^{\min}, p_1^*\} | w) - F(p_1^{\min} | w)) \\ &- \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \end{aligned} \quad 4-4-12$$

All $p_i^* \leq p_1^U$, so substituting $p_1^{\min} = p_1^U$ into (4-4-12) gives

$$H(p_1^U | w) = 1 - (F(p_1^U | w) - F(p_1^U | w)) - \sum_{j=2}^{\infty} (F(p_1^U | w) - F(p_1^U | w)) \prod_{i=1}^{j-1} (1 - F(p_1^U | w)) \quad 4-4-13$$

= 1

Q.E.D.

The c.d.f. of minimum observed prices should have the property that, if search is always of a length of exactly one observation on p_1 , it should coincide with the c.d.f. of selling prices $F(p_1 | w)$.

Remark 4-4-4:

$$\Pr(\psi_1^* = 1 | w) = 1 \quad \text{if and only if } p_1^* = p_1^U$$

Proof:

The proof is immediate upon substituting $p_1^* = p_1^U$ into $\Pr(\psi_j^* = 1 | w)$ for all $j \geq 1$, given by Proposition 4-3-2.

Q.E.D.

Remark 4-4-5:

$H(p_1^{\min} | w) \equiv F(p_1^{\min} | w)$ if and only if search conditions are such that $p_1^* = p_1^U$.

Proof:

First it is shown that if $p_1^* = p_1^U$, then $H(p_1^{\min}|w) \equiv F(p_1^{\min}|w)$.

Second it is shown that if $H(p_1^{\min}|w) \equiv F(p_1^{\min}|w)$, then $p_1^* = p_1^U$.

(i) Substituting $p_1^* = p_1^U$ into (4-4-12) gives

$$H(p_1^{\min}|w) = 1 - (F(p_1^U|w) - F(p_1^{\min}|w)) - \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\}|w) - F(p_1^{\min}|w)) \prod_{i=2}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\}|w)) (1 - F(p_1^U|w)) \quad 4-4-14$$

for any $p_1^{\min} \in [p_1^L, p_1^U]$, $F(p_1^U|w) = 1$ so

$$H(p_1^{\min}|w) = 1 - (1 - F(p_1^{\min}|w)) = F(p_1^{\min}|w) \quad 4-4-15$$

for any $p_1^{\min} \in [p_1^L, p_1^U]$ ie, $H(p_1^{\min}|w) \equiv F(p_1^{\min}|w)$.

(ii) If $H(p_1^{\min}|w) \equiv F(p_1^{\min}|w)$, then, from (4-4-12),

$$F(p_1^{\min}|w) \equiv 1 - (F(\max\{p_1^{\min}, p_1^*\}|w) - F(p_1^{\min}|w)) - \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\}|w) - F(p_1^{\min}|w)) \prod_{i=2}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\}|w)) \quad 4-4-16$$

$$= \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\}|w) - F(p_1^{\min}|w)) \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\}|w))$$

Clearly equivalence (4-4-16) is true if and only if

$$0 \equiv 1 - F(\max\{p_1^{\min}, p_1^*\}|w) - \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\}|w) - F(p_1^{\min}|w)) \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\}|w)) \quad 4-4-17$$

Equivalence (4-4-17) is true if and only if $p_1^* = p_1^U$.

Q.E.D.

The driving force for systematic consumer search of more than one observation on p_1 is the belief that, on average, this will provide a searcher with a lower selling price for commodity 1 than would be obtained on average from a random search of just one observation on

p_1 . Stigler showed this to be the case for his version of the search problem. Previous research on the sequential search problem has asserted, but not proved, that the mean $E_h[p_1^{\min}|w]$ of the p.d.f. of minimum observed prices, $h(p_1^{\min}|w)$, must be no greater than the mean selling price $E_f[p_1|w]$, where

$$h(p_1^{\min}|w) = \partial H(p_1^{\min}|w) / \partial p_1^{\min} \quad 4-4-18$$

$$E_h[p_1^{\min}|w] = \int_{p_1^L}^{p_1^U} p_1^{\min} h(p_1^{\min}|w) dp_1^{\min} \quad 4-4-19$$

$$E_f[p_1|w] = \int_{p_1^L}^{p_1^U} p_1 f(p_1|w) dp_1 \quad 4-4-20$$

Lemma 4-4-6 derives a property of the cumulative p.d.f. of minimum observed prices which is used in Proposition 4-4-8 to prove the assertion that $E_h[p_1^{\min}|w] \leq E_f[p_1|w]$.

The more frequently searchers extend their searches to two or more observations on p_1 , the more frequently selling prices below the mean selling price $E_f[p_1|w]$ will be observed, and the lower will be the mean minimum observed selling price $E_h[p_1^{\min}|w]$. Accordingly, any change in search costs which increases the mean search length, should also decrease the mean minimum observed selling price. This too is a common unproved assertion among previous authors. A proof is given by Proposition 4-4-7.

A third frequent assertion may be paraphrased as "increased search decreases price dispersion". Stigler's seminal paper contains the statement that "Since the variance of the expected minimum price decreases with additional search, the prices paid by inexperienced

buyers will also have a larger variance" [50, p. 219]. Axell's empirical results [1, p. 85-90] indicate that both the mean and variance of the p.d.f. of minimum observed prices are increased by an increase in the marginal financial cost of search. Rothschild seems to sum up the intuition of previous writers when he says "It should turn out in most sensible models that increased search activity will decrease price dispersion" [43, p. 692]. The measures of price dispersion quoted are either the variance or the coefficient of variation of the p.d.f. of minimum observed prices. What has been overlooked, however, is that the change in variance, or coefficient of variation, caused by a change in a marginal search cost, depends upon the form of the p.d.f. of selling prices $f(p_1|w)$. For instance, the assertion is true if $f(p_1|w)$ is uni-modal and either symmetric or skewed to the right. However, if $f(p_1|w)$ is skewed to the left, it is quite possible for the variance or coefficient of variation of the p.d.f. of minimum observed prices to be increased by a decrease in a marginal search cost. The derivative of the variance of the p.d.f. of minimum observed prices w.r.t. the k th marginal search cost c_k is presented in Remark 4-4-9, along with a simple numerical example in substantiation of the claim that these measures of price dispersion may be increased by an increase in a marginal search cost.

Lemma 4-4-6:

$$\frac{\partial}{\partial c_k}(H(p_1^{\min}|w)) \leq 0, \text{ for all } k \geq 1, \text{ and for all but a finite}$$

number of values of $p_1^{\min} \in [p_1^L, p_1^U]$ for which left and right hand derivatives differ but are non-positive and exist.

Proof:

From Proposition 4-4-1, $\frac{\partial}{\partial c_k}(H(p_1^{\min}|w))$

$$\begin{aligned}
&= -f(\max\{p_1^{\min}, p_1^*\} | w) \cdot \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_1^*\}) - \sum_{j=2}^{\infty} \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \cdot \\
&\quad f(\max\{p_1^{\min}, p_j^*\} | w) \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_j^*\}) \\
&+ (F(\max\{p_1^{\min}, p_2^*\} | w) - F(p_1^{\min} | w)) \cdot f(\max\{p_1^{\min}, p_1^*\} | w) \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_1^*\}) \quad 4-4-21 \\
&+ \sum_{j=3}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \sum_{\substack{n=1 \\ i \neq n}}^{j-1} \prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \\
&\quad f(\max\{p_1^{\min}, p_n^*\} | w) \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_n^*\}) \\
&= f(\max\{p_1^{\min}, p_1^*\} | w) \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_1^*\}) [-1 + (F(\max\{p_1^{\min}, p_2^*\} | w) - F(p_1^{\min} | w)) \\
&+ \sum_{j=3}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \prod_{i=2}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w))] \\
&+ \sum_{n=2}^{\infty} f(\max\{p_1^{\min}, p_n^*\} | w) \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_n^*\}) [- \prod_{i=1}^{n-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \quad 4-4-22 \\
&+ \sum_{j=n+1}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \prod_{\substack{i=1 \\ i \neq n}}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w))]
\end{aligned}$$

Note that (i) $\frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_n^*\}) = 0$, for all $k > n+1$, by Lemma 4-3-4.

$$\begin{aligned}
\text{(ii)} \quad \frac{\partial}{\partial c_k} (\max\{p_1^{\min}, p_n^*\}) &= 0, \text{ if } p_1^{\min} > p_n^* \\
&\geq 0, \text{ if } p_1^{\min} < p_n^*, \text{ by Lemma 4-3-4}
\end{aligned}$$

and is undefined if $p_1^{\min} = p_n^*$.

Hence the derivative $\frac{\partial}{\partial c_k} (H(p_1^{\min} | w))$ will be undefined for all values of $p_1^{\min} = p_n^*$, for all $n = 1, 2, \dots$. Since (4-2-2) guarantees search is always of finite length, $H(p_1^{\min} | w)$ will depend upon only a finite number of p_n^* , so the result will hold for all but the finite number

of values of $p_1^{\min} = p_n^* \in [p_1^L, p_1^U]$. To complete the proof consider the n th term

$$\begin{aligned}
 & [- \prod_{i=1}^{n-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w)) + \sum_{j=n+1}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \\
 & \prod_{i=1}^{j-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w))] \\
 & i \neq n \\
 & = \prod_{i=1}^{n-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w)) [-1 + F(\max\{p_1^{\min}, p_{n+1}^*\} | w) - F(p_1^{\min} | w) \\
 & + \sum_{j=n+2}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \prod_{i=n+1}^{j-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w))] \quad 4-4-23
 \end{aligned}$$

$$\begin{aligned}
 & = \prod_{i=1}^{n-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w)) [-1 + (1-F(p_1^{\min} | w)) - (1-F(\max\{p_1^{\min}, p_{n+1}^*\} | w)) \quad 4-4-24 \\
 & + \sum_{j=n+2}^{\infty} ((1-F(p_1^{\min} | w)) - (1-F(\max\{p_1^{\min}, p_j^*\} | w))) \prod_{i=n+1}^{j-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w))]
 \end{aligned}$$

(4-2-2) guarantees that eventually a $p_n^* = p_1^U$. Suppose $p_j^* = p_1^U$. Then

$$(1-F(\max\{p_1^{\min}, p_j^*\} | w)) = (1-F(p_1^U | w)) = 0 \quad 4-4-25$$

and (4-4-24) may be rewritten as

$$\begin{aligned}
 & \prod_{i=1}^{n-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w)) [(-1 + 1-F(p_1^{\min} | w)) + (-1 + 1-F(p_1^{\min} | w)) \\
 & (1-F(\max\{p_1^{\min}, p_{n+1}^*\} | w)) \\
 & + \dots + (-1 + 1-F(p_1^{\min} | w)) \prod_{i=n+1}^{J-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w))] \quad 4-4-26
 \end{aligned}$$

$$\begin{aligned}
 & = \prod_{i=1}^{n-1} (1-F(\max\{p_1^{\min}, p_i^*\} | w)) (-F(p_1^{\min} | w)) (1 + \sum_{j=n+1}^{J-1} \prod_{i=n+1}^j (1-F(\max\{p_1^{\min}, p_i^*\} | w))) \\
 & \leq 0 \quad 4-4-27
 \end{aligned}$$

$$4-4-28$$

Similarly, the first summation terms of (4-4-22) can be shown to be

$$[-1 + (F(\max\{p_1^{\min}, p_2^*\} | w) - F(p_1^{\min} | w)) + \sum_{j=3}^{\infty} (F(\max\{p_1^{\min}, p_j^*\} | w) - F(p_1^{\min} | w)) \prod_{i=2}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w))] \quad 4-4-29$$

$$= -1 + (1 - F(p_1^{\min} | w)) + (1 - F(\max\{p_1^{\min}, p_2^*\} | w)) + \sum_{j=3}^{\infty} ((1 - F(p_1^{\min} | w)) - (1 - F(\max\{p_1^{\min}, p_j^*\} | w))) \prod_{i=2}^{j-1} (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \quad 4-4-30$$

$$= -F(p_1^{\min} | w) \sum_{j=2}^{\infty} \prod_{i=2}^j (1 - F(\max\{p_1^{\min}, p_i^*\} | w)) \quad 4-4-31$$

$$\leq 0 \quad 4-4-32$$

Comparing (4-4-28) and (4-4-32) to (4-4-22) shows each of the terms of (4-4-22) is non-positive. Therefore $\frac{\partial}{\partial c_k}(H(p_1^{\min} | w)) \leq 0$ for all $k \geq 1$ and for all values of $p_1^{\min} \in [p_1^L, p_1^U]$ except the values of $p_1^{\min} = p_n^*$ for $n = 1, 2, \dots, J-1$ where the left hand derivatives are negative and the right hand derivatives are zero. Q.E.D.

Proposition 4-4-7:

$$\frac{\partial}{\partial c_k}(E_h[p_1^{\min} | w]) = 0, \text{ if } k = 1. \\ \geq 0, \text{ if } k > 1.$$

Proof:

$$E_h[p_1^{\min} | w] = \int_{p_1^L}^{p_1^U} p_1^{\min} h(p_1^{\min} | w) dp_1^{\min} \quad 4-4-33$$

$$\frac{\partial}{\partial c_k}(E_h[p_1^{\min} | w]) = \int_{p_1^L}^{p_1^U} p_1^{\min} \frac{\partial h(p_1^{\min} | w)}{\partial c_k} dp_1^{\min} \quad 4-4-34$$

The result of Lemma 3-5-1 may be applied to show

$$\frac{\partial}{\partial c_k}(E_h[p_1^{\min}|w]) \geq 0 \text{ by noting (i) that if } g(p_1^{\min}) = p_1^{\min},$$

$$\text{then } \frac{\partial g(p_1^{\min})}{\partial p_1^{\min}} = 1 > 0 \quad 4-4-35$$

(ii) the result of Lemma 4-4-6 that

$$\frac{\partial}{\partial c_k}(H(p_1^{\min}|w)) \leq 0 \text{ for all } p_1^{\min} \in [p_1^L, p_1^U]$$

except for a finite number of values of p_1^{\min} .

Q.E.D.

This result is a theoretical verification of the result inferred by Axell, Telser and Gastwirth from their numerical studies.

Proposition 4-4-8:

$$E_f[p_1|w] \geq E_h[p_1^{\min}|w]$$

Proof:

Remark 4-4-5 shows $H(p_1^{\min}|w) \equiv F(p_1|w)$ if and only if $p_1^* = p_1^U$.

Hence, $E_f[p_1|w] = E_h[p_1^{\min}|w]$ if and only if $p_1^* = p_1^U$

p_1^* is a function of c_2 alone and Lemma 4-3-4 shows

$$\frac{\partial p_1^*}{\partial c_2} \geq 0 \quad 4-4-36$$

Consider a decrease in c_2 such that now $p_1^* < p_1^U$. Then, by Remark 4-4-5 and Proposition 4-4-7,

$$E_f[p_1|w] > E_h[p_1^{\min}|w] \quad 4-4-37$$

Hence, for any sequence of marginal financial search costs,

$$E_f[p_1|w] \geq E_h[p_1^{\min}|w] \quad 4-4-38$$

Q.E.D.

Remark 4-4-9:

$$\frac{\partial}{\partial c_k}(\text{var}_h(p_1^{\min}|w)) = \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w])^2 \frac{\partial h(p_1^{\min}|w)}{\partial c_k} dp_1^{\min}$$

Proof:

$$\text{var}_h(p_1^{\min}|w) = \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w])^2 \cdot h(p_1^{\min}|w) dp_1^{\min} \quad 4-4-39$$

$$\frac{\partial}{\partial c_k}(\text{var}_h(p_1^{\min}|w)) = \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w])^2 \cdot \frac{\partial h(p_1^{\min}|w)}{\partial c_k} \cdot dp_1^{\min} \quad 4-4-40$$

$$- 2 \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w]) \frac{\partial E_h[p_1^{\min}|w]}{\partial c_k} \cdot h(p_1^{\min}|w) dp_1^{\min}$$

$E_h[p_1^{\min}|w]$ is not a function of p_1^{\min} so

$$\begin{aligned} & \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w]) \frac{\partial E_h[p_1^{\min}|w]}{\partial c_k} h(p_1^{\min}|w) dp_1^{\min} \\ &= \frac{\partial E_h[p_1^{\min}|w]}{\partial c_k} \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w]) h(p_1^{\min}|w) dp_1^{\min} \quad 4-4-41 \end{aligned}$$

$$= \frac{\partial E_h[p_1^{\min}|w]}{\partial c_k} \left\{ \int_{p_1^L}^{p_1^U} p_1^{\min} \cdot h(p_1^{\min}|w) dp_1^{\min} - E_h[p_1^{\min}|w] \right\} \quad 4-4-42$$

$$= 0 \quad 4-4-43$$

Substituting (4-4-43) into (4-4-40) gives

$$\frac{\partial}{\partial c_k}(\text{var}_h(p_1^{\min}|w)) = \int_{p_1^L}^{p_1^U} (p_1^{\min} - E_h[p_1^{\min}|w])^2 \frac{\partial h(p_1^{\min}|w)}{\partial c_k} dp_1^{\min} \quad 4-4-44$$

Q.E.D.

Consider (4-4-44). $\partial h(p_1^{\min}|w)/\partial c_k$ will be positive for some values of p_1^{\min} and negative for others. Whether or not $\partial(\text{var}_h(p_1^{\min}|w))/\partial c_k$ is positive or negative depends on the relative weightings given to the positive and negative regions of $\partial h(p_1^{\min}|w)/\partial c_k$ by the corresponding values of $(p_1^{\min} - E_h[p_1^{\min}|w])^2$. In any event, it is definitely not the case that the variance of the p.d.f. of minimum observed prices must always be decreasing w.r.t. any of the marginal financial search costs. A simple numerical counter-example is now presented.

Suppose there are only two selling prices in the market, \$10 and \$20, with respective probabilities of 0.2 and 0.8, ie.

$$f(p_1|w) = \begin{cases} 0.2, & p_1 = \$10 \\ 0.8, & p_1 = \$20 \end{cases} \quad 4-4-45$$

Also suppose that the searcher's prior p.d.f. $g(w)$ and the costs of his chosen search path are such that $p_1^* = p_2^* = \$20$. Then he will only take one observation on p_1 and his ex ante probabilities for minimum observed prices of \$10 and \$20 are

$$h(p_1^{\min}|w) = \begin{cases} 0.2, & p_1^{\min} = \$10 \\ 0.8, & p_1^{\min} = \$20 \end{cases} \quad 4-4-46$$

The mean, variance and coefficient of variation for this distribution are

$$E_h[p_1^{\min}|w] = 18.0 \quad 4-4-47$$

$$\text{var}_h(p_1^{\min}|w) = 16.0 \quad 4-4-48$$

$$C.V. = 0.222 \quad 4-4-49$$

Now suppose the second marginal financial search cost c_2 is lowered so that $p_1^* = \$15$ but $p_2^* = \$20$ still. The searcher's ex ante probabilities for minimum observed prices of \$10 and \$20 are now

$$h(p_1^{\min}|w) = \begin{cases} 0.36, & p_1^{\min} = \$10 \\ 0.64, & p_1^{\min} = \$20 \end{cases} \quad 4-4-50$$

The mean, variance and coefficient of variation for this distribution are

$$E_h[p_1^{\min}|w] = 16.4 \quad 4-4-51$$

$$\text{var}_h(p_1^{\min}|w) = 23.04 \quad 4-4-52$$

$$C.V. = 0.293 \quad 4-4-53$$

Comparing (4-4-47) to (4-4-51), (4-4-48) to (4-4-52) and (4-4-49) to (4-4-53) shows the decrease in c_2 has caused the mean minimum observed price to fall and both the variance and coefficient of variation of the distribution of minimum observed prices to rise. By either criterion, therefore, the decrease in c_2 has caused an increase in price dispersion.

This section has established the p.d.f. of minimum observed prices and examined some of its properties. The model used was a sequential search model in which myopic search was optimal and in which the searcher learned about a partially unknown p.d.f. of selling

prices. The next section treats the simpler case in which the form and parameters of the p.d.f. of selling prices are known completely to the searcher. Following this, Section 4-6 discusses the relationship of the p.d.f. of minimum observed prices to the p.d.f. of transaction prices.

SECTION 4-5: THE EX ANTE PROBABILITY FUNCTIONS OF SEARCH LENGTHS AND MINIMUM OBSERVED PRICES FOR A MYOPIC SEQUENTIAL SEARCH IN WHICH THE PROBABILITY DENSITY FUNCTION OF SELLING PRICES IS KNOWN

In this section it is assumed that w is known to the searcher. This provides the searcher with complete information about the functional form and values of conditioning parameters of the p.d.f. of selling prices $f(p_1|w)$. The additional information simplifies the searcher's task, and the analysis, considerably because there is no longer a need to glean information from observed prices as to the likely values of the parameters w conditioning $f(p_1|w)$. Recall from (4-2-10) that, in search models where w is not known to the searcher, reservation prices are functions of observed prices, because the values of observed prices alter the searcher's perception of the p.d.f. of selling prices facing him. If $f(p_1|w)$ is completely known, however, the reservation prices are only functions of the marginal financial search costs, since the searcher's perception of $f(p_1|w)$ is perfect and unchanging at each stage of his search. By analogy to (4-2-6), therefore, the searcher's j th reservation price is $C_j = C_j(c_{j+1})$ 4-5-1

where

$$(i) \quad \int_{p_1^L}^{C_j} (C_j - p_1) f(p_1|w) dp_1 = c_{j+1}, \text{ if } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f(p_1|w) dp_1 > c_{j+1}$$

4-5-2

$$(ii) \quad C_j = p_1^U, \text{ if } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f(p_1 | w) dp_1 \leq c_{j+1}$$

The following analysis first of all shows C_j is strictly increasing w.r.t. c_{j+1} alone. Corollary 4-5-1, which proves this, is a special case of Lemma 4-2-1 which proves the corresponding result when reservation prices are functions of observed prices as well. Corollary 4-5-1 is presented merely for the reader's convenience. The conditions assumed in order to guarantee the optimality of myopic search also guarantee the optimal search path is characterised by a monotonic non-decreasing sequence of marginal financial search costs. Hence, the searcher will possess a monotonic non-decreasing sequence of reservation prices. This monotonicity allows the expressions for the ex ante probability functions of search lengths and minimum observed prices to be greatly simplified.

Corollary 4-5-1:

$$\frac{\partial C_j}{\partial c_{j+1}} \left\{ \begin{array}{l} \geq 1 \\ = 0 \\ \text{is undefined} \end{array} \right\} \text{ as } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f(p_1 | w) dp_1 \left\{ \begin{array}{l} > c_{j+1} \\ < c_{j+1} \\ = c_{j+1} \end{array} \right.$$

Proof:

$$(i) \quad \text{If } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f(p_1 | w) dp_1 < c_{j+1} \text{ then, by (ii) of (4-5-2),}$$

$$\frac{\partial C_j}{\partial c_{j+1}} = \frac{\partial p_1^U}{\partial c_{j+1}} = 0$$

4-5-3

$$(ii) \quad \text{If } \int_{p_1^L}^{p_1^U} (p_1^U - p_1) f(p_1 | w) dp_1 > c_{j+1} \text{ then, by (i) of (4-5-2),}$$

there is a value $C_j < p_1^U$ of C_j such that

$$\int_{p_1^L}^{C_j} (C_j - p_1) f(p_1 | w) dp_1 = c_{j+1} \quad 4-5-4$$

Totally differentiating (4-5-4) w.r.t. c_{j+1} gives

$$\frac{\partial C_j}{\partial c_{j+1}} \int_{p_1^L}^{C_j} f(p_1 | w) dp_1 = 1 \quad 4-5-5$$

$$\text{so} \quad \frac{\partial C_j}{\partial c_{j+1}} = \left(\int_{p_1^L}^{C_j} f(p_1 | w) dp_1 \right)^{-1} \geq 1 \quad 4-5-6$$

(iii) The non-smoothness of C_j for c_{j+1} such that

$$\int_{p_1^L}^{p_1^U} (p_1^U - p_1) f(p_1 | w) dp_1 = c_{j+1}$$

is implied by (i) and (ii).

Q.E.D.

Proposition 4-5-2:

If $f(p_1 | w)$ is known, myopic optimal sequential search is characterised by a monotonic increasing sequence of the smallest reservation prices C_j .

Proof:

Proposition 3-4-1 shows that, under assumptions (4-2-1), (1-4-28) and (3-5-1), the optimal search path is such that $c_i = c_{(i)}$ for all $i = 1, 2, \dots$ ie. the marginal financial search costs associated with the optimal search path form a monotonic increasing sequence of the smallest marginal financial search costs. The desired result is therefore given immediately by Corollary 4-5-1.

Q.E.D.

The ex ante probability of a searcher, who does not know w exactly, stopping after j observations on p_1 , depends upon the values of all of p_1^*, \dots, p_j^* (see Proposition 4-3-2). The simplification obtained by applying the result of Proposition 4-5-2 is that the ex ante probability of a searcher, who has complete knowledge of $f(p_1|w)$, stopping after j observations on p_1 , depends only upon the values of C_{j-1} and C_j . The ex ante p.m.f. of search lengths and the ex ante p.d.f. of minimum observed prices for such a searcher are presented in Corollary 4-5-3 and Corollary 4-5-4.

Corollary 4-5-3:

If $f(p_1|w)$ is known, the ex ante probability mass function of search lengths is

$$\Pr(\psi_1^* = 1|w) = F(C_1|w)$$

and

$$\Pr(\psi_j^* = 1|w) = (1 - F(C_{j-1}|w))^{j-1} + (1 - F(C_j|w))^j \text{ for all } j \geq 2.$$

Proof:

First consider $j=1$. From (4-3-36),

$$\begin{aligned} \Pr(\psi_1^* = 1|w) &= \Pr(p_{11}^{\min} \leq C_1|w) = \Pr(p_1^1 \leq C_1|w) \\ &= F(C_1|w). \end{aligned} \quad 4-5-7$$

Now consider $j \geq 2$. From (4-3-38),

$$\Pr(\psi_j^* = 1|w) = \Pr(p_{1i}^{\min} > C_i \text{ for all } i=1, \dots, j-1|w) - \Pr(p_{1i}^{\min} > C_i \text{ for all } i=1, \dots, j|w) \quad 4-5-8$$

From (4-3-39), $\Pr(p_{1i}^{\min} > C_i \text{ for all } i=1, \dots, j|w)$

$$= \Pr(p_1^1 > C_1, \min(p_1^1, p_1^2) > C_2, \dots, \min(p_1^1, \dots, p_1^j) > C_j|w) \quad 4-5-9$$

$$= \Pr(p_1^1 > \max\{C_1, \dots, C_j\}, p_1^2 > \max\{C_2, \dots, C_j\}, \dots, p_1^j > C_j|w) \quad 4-5-10$$

$$\text{or } p_{1j}^{\min} > p_1^{\min} \text{ and } p_1^1 > c_1, \dots, p_1^{j-1} > c_{j-1}, p_1^j \leq c_j$$

$$\dots | w)$$

$$= \Pr(p_1^{\min} < p_1^1 \leq c_1 | w)$$

$$+ \sum_{j=2}^{\infty} \Pr(p_1^1 > p_1^{\min}, \dots, p_1^j > p_1^{\min}, p_1^1 > c_1, \dots, p_1^{j-1} > c_{j-1}, p_1^j \leq c_j | w)$$

4-5-16

$$= \Pr(p_1^{\min} < p_1^1 \leq c_1 | w)$$

$$+ \sum_{j=2}^{\infty} \Pr(p_1^1 > \max\{p_1^{\min}, c_1\}, \dots, p_1^{j-1} > \max\{p_1^{\min}, c_{j-1}\}, p_1^{\min} < p_1^j \leq c_j | w)$$

4-5-17

$$= F(\max\{p_1^{\min}, c_1\} | w) - F(p_1^{\min} | w) + \sum_{j=2}^{\infty} (F(\max\{p_1^{\min}, c_j\} | w) - F(p_1^{\min} | w)).$$

$$\prod_{i=1}^{j-1} (1 - F(\max\{p_1^{\min}, c_i\} | w))$$

4-5-18

Rearranging (4-5-18) gives the desired result.

Q.E.D.

SECTION 4-6: THE RELATIONSHIP OF THE PROBABILITY DENSITY FUNCTION OF MINIMUM OBSERVED PRICES TO THE PROBABILITY DENSITY FUNCTION OF TRANSACTION PRICES AND PRICE DISPERSION

The analytic thrust of search models is towards a description of the attainment and maintenance of a market equilibrium in which prices are dispersed. The distribution of transaction prices, and the corresponding distribution of quantities purchased, are therefore desirable goals for search theorists since they will provide understanding of the nature of such an equilibrium. This chapter provides one such distribution of transaction prices, but only under a set of

severely restrictive conditions, the foremost of which is the restriction that the demand for commodity 1 is always one unit; $x_1^* \equiv 1$. It is possible to argue that such a restriction is not unreasonable in a real world frequently characterised by indivisibilities but such an argument must be supplemented by a restriction that the commodity is a necessity and that the marginal utility of a second unit of the commodity is very small (or negative) when compared to the marginal utility of the first unit. Only in such cases can we be assured that the "realism" of the restriction $x_1^* \equiv 1$ is "reasonable". A more general approach than the one used in this chapter will be necessary to generate a distribution of transaction prices with more general applicability.

One trivial generalisation is to the problem of bulk buying. It was stated in Chapter 1 that the severity of the assumption that $x_1^* \equiv 1$ was due to x_1^* being independent of prices and wealth and that x_1^* being identically unity in particular was merely a matter of an appropriate choice of units. If the single unit demanded is actually m units of another size, then all the preceding analysis of Chapters 2, 3 and 4 may be viewed as solving the problem of a search for the best possible purchase conditions for a bulk order of m units of commodity 1.

Adapting the analysis of this chapter to variable integer demands for commodity 1 does not at first appear to be a substantially more difficult problem than the one solved in this chapter. Unfortunately, removing the condition $x_1^* \equiv 1$ also removes one of the conditions necessary for the use of minimisation of total expected expenditure on commodity 1 as a valid proxy for consumer utility maximisation. The analysis required for developing more general distributions of

transaction prices based on optimal search behaviour will have to be carried out in utility terms, rather than financial terms, and consequently will be more difficult.

The p.d.f. of transaction prices, by itself, is an adequate measure of price dispersion only if each transaction is an exchange of the same quantity. If the amount exchanged varies from one transaction to another, then, to completely characterise the extent of price dispersion in a market, the distribution of transaction prices must be supplemented with the quantities exchanged at each transaction price. Compare two markets. 100 transactions are carried out in a given time period in each market. In both markets 50 of these transactions are carried out at a transaction price of \$10 and the remaining 50 transactions at a price of \$20. In the first market, all transactions are unit exchanges. In the second market, all transactions at a price of \$20 are unit exchanges while all transactions at a price of \$10 are two unit exchanges. Clearly price dispersion measured on a per transaction basis is different to price dispersion measured on a per unit exchanged basis. To distinguish between these two measures of dispersion, a joint consideration of the p.d.f.'s of demands and transaction prices is required.

The literature already contains some discussion about the behaviour of price dispersion in the long-term. The conclusions offered vary widely, principally because the models from which these conclusions are drawn vary widely in their structures. Even so, Fisher's model [10] leads to the conclusion that any initial price dispersion eventually collapses to the perfectly competitive price while Hey's model [15], which is not too dissimilar in structure to

Fisher's model, leads to the conclusion that any initial price dispersion eventually collapses to the monopoly price. While they may not agree about the price to which this collapse converges, almost all search models in which market conditions are unchanging predict that initial price dispersion eventually vanishes. This conclusion is what such models should produce, since they consider a possibly infinite time horizon within which the same market agents may continue to amass information. If, as Stigler put it, "Price dispersion is a manifestation - ... - of ignorance in the market" [50, p. 214] then these agents have an infinitely long period in which to reduce their stocks of uncertainty to zero and remove any price dispersion.

Dynamic models permit changing market conditions and so produce more realistic conclusions. In Mortensen's model [33] a wages distribution does not collapse with the passage of time. Instead it varies in response to variations in the actual and expected rates of wage inflation, the rate of product price inflation, the unemployment rate and the rate of adjustment of expectations. Ioannides' model [18], which also shows a persistent price dispersion, comes closest of all to giving meaning to the term "equilibrium" in the context of price dispersion. He recognises that equilibrium requires equality between the rate at which sellers supply goods and the rate at which buyers purchase the goods, and that the distributions of selling prices and buyers' reservation prices must be stationary. The far more realistic conclusion reached is that a major source of price dispersion is the continuous turnover in market participants, thus maintaining a stock of ignorance about prices in the market.

One may reasonably ask if it is worthwhile to pursue the argument over the long-term behaviour of price dispersion in models which assume unchanging market conditions. After all, many market conditioning factors are beyond our control. For example, a drought on the Canadian Prairies will cause an upward shift in the distribution of wheat prices. Such factors cause changes to a market which can be learnt about only slowly by the majority of the agents in the market. The changes therefore depreciate the stock of knowledge of the market that the market's agents possess. Consequently these factors are a source of price dispersion. To ignore them, however, is useful if it assists in gaining an understanding of the processes by which markets approach equilibrium, and in understanding what influences the rates of these processes. This understanding will contribute towards understanding market reactions as changes in market conditions alter the position of the equilibrium to which the market strives.

CHAPTER V

AN INDIRECT UTILITY FUNCTION APPROACH TO SEARCH PROBLEMS[†]SECTION 5-1: THE APPROPRIATENESS OF USING THE INDIRECT UTILITY
FUNCTION IN SEARCH PROBLEMS

Many of the previous analyses of consumer search problems have been carried out in financial terms. Considerable attention was given in Chapters 2, 3 and 4 to determining when such analyses could be considered equivalent to the more familiar neo-classical problem of a consumer maximising his utility while restricted by constraints on the amounts he can consume. The conclusion reached was that the problems were equivalent if psychic search costs did not exist and if the consumer's utility function belonged to a severely restricted set Ω defined by (2-7-14). In Section 1-4 the restriction of a fixed demand for the searched for commodity was shown to make all consumer preference pre-orderings other than those with a particular degree of non-smoothness inadmissible for these consumer search problems. Together, these restrictions show that only a very heavily restricted set of consumer preference pre-orderings is admissible to consumer search problems analysed in financial terms. In addition, psychic search costs undoubtedly are a large component, sometimes the only component, of a searcher's overall search cost. The rationale for carrying out solely financial analyses of search problems is presumably that the generality lost is compensated for by the analytic ease gained. The loss in generality has been shown to be great. This chapter shows the additional difficulties encountered in a cardinal utility analysis of consumer decision-making in search

†: Much of the material in this chapter was presented by Manning and Morgan [29] to the Fifth Conference of Economists, Brisbane, Australia, in August 1975.

problems are quite surmountable.

The principal motive for analysing consumer search problems in financial terms appears to be that prices and consumer wealth are variable. The usual neo-classical analysis, conducted in direct utility terms, takes these parameters as given and fixed, but this is only because the usual objective of a neo-classical consumer problem is a specification of the consumer's demands. The dual to this problem is the minimisation of the consumer's indirect utility function, subject to a constraint or constraints in which the consumer's quantities demanded are taken as given and fixed. The indirect utility function contains exactly the same information about the consumer's preferences as the direct utility function. Which function is chosen depends only upon the analytic objective which one has in mind. If the objective is the specification of consumer demands corresponding to given prices and wealth, then the analysis is conducted more easily with the direct utility function since the demands are arguments of this function. If the objective is the specification of prices and wealth corresponding to given demands, then the analysis is conducted more easily with the indirect utility function since prices and wealth are arguments of this function. Hotelling [16, p. 583-597] made exactly this distinction in 1932 in a paper which used pricing functions, as opposed to demand functions, to resolve a paradox put forward by Edgeworth. Later Roy [44], [45] specified the dual consumer problem and Houthakker [17] gave the name of "indirect utility function" to the objective function of this problem.

The duality of the direct and indirect utility functions has been shown to be complete, each possessing all the general properties of the other (see Samuelson [39] and Haque [14]). The reader is

referred to these works, to Sato [49] and to Hadar [13, p. 197-208] for discussions on the properties of the indirect utility function. The main point for us to realise is that, as Houthakker says with regard to the use of the direct or indirect utility functions, "The choice between them is, therefore, a matter of mathematical convenience, at any rate in one-person problems" [17, p. 158, footnote 1]. The variables of a consumer search problem are usually prices and wealth. The indirect utility function is, therefore, usually the most appropriate objective function to use in the analysis of consumer search problems.

SECTION 5-2: NOTATION AND ASSUMPTIONS

The search models examined in Chapters 2, 3 and 4 were all assumed to be such that sequential search behaviour was optimal. In this chapter it is assumed that the conditions of search are such that Stigleresque search is optimal. A reason for this is that, until now, the thesis has not specifically considered Stigleresque search. The indirect utility function approach is equally useful in analysing sequential search problems (see Chapter 2 and Kohn and Shavell [20], and Chapter 6 of this thesis which includes sequential search as a special case of an indirect utility function approach to generalised search).

Assumption: The nature of the search problem is such that

Stigleresque search is optimal.

5-2-1

The problem faced by the Stigleresque searcher is the ex ante choice of the number of observations on p_1 which he expects to maximise his ex ante utility. Recall from (1-4-23) that Stigleresque search

persists only through the first period T_1 and that any number of observations on p_1 can be made in this period. The number of observations on p_1 decided upon at the beginning of period T_1 is not varied once search has begun. The marginal search costs incurred by the searcher when taking his i th observation on p_1 are, from Section 1-4, a financial cost of c_{1i} and a psychic cost of K_{1i} .

Assumption: The total psychic cost of taking j observations on p_1 in period T_1 is

$$K_1(j) = \sum_{i=1}^j K_{1i} \quad 5-2-2$$

The searcher's wealth at the end of period T_1 , after j observations have been taken on p_1 , is, from (1-4-18),

$$\bar{M}_1 = M - c(1) = M - \sum_{i=1}^j c_{1i} \quad 5-2-3$$

\bar{M}_1 and $c(1)$ will be written as $\bar{M}_1(j)$ and $c_1(j)$ to denote their dependencies on j , ie.

$$\bar{M}_1(j) = M - c_1(j) = M - \sum_{i=1}^j c_{1i} \quad 5-2-4$$

In Appendix 4, the p.d.f. of the minimum of a sample of j independent price observations drawn from a population with p.d.f. $f(p_1|w)$ is shown to be

$$h(p_1^{\min}|j,w) = j(1-F(p_1^{\min}|w))^{j-1}f(p_1^{\min}|w) \quad 5-2-5$$

for all $p_1^{\min} \in [p_1^L, p_1^U]$. The utility the searcher expects ex ante to enjoy at the end of period T_1 is thus, by (1-4-11),

$$E[I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1)|j,w] = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) h(p_1^{\min}|j,w) dp_1^{\min} \quad 5-2-6$$

The searcher's indirect utility function is assumed to possess the following properties. In Section 5-3, these properties are shown to be sufficient for the indirect utility function to be strictly concave with respect to j .

$$\frac{\partial I}{\partial p_i} < 0 \quad \text{for all } i = 1, \dots, \ell \quad 5-2-7$$

$$\frac{\partial^2 I}{\partial p_i^2} > 0 \quad \text{for all } i = 1, \dots, \ell \quad 5-2-8$$

ie. the marginal indirect utility of the i th price is negative and diminishing

$$\frac{\partial I}{\partial M_1} > 0 \quad 5-2-9$$

$$\frac{\partial^2 I}{\partial M_1^2} < 0 \quad 5-2-10$$

ie. the marginal indirect utility of income is positive and diminishing

$$\frac{\partial^2 I}{\partial M_1 \partial p_i} < 0 \quad \text{for all } i = 1, \dots, \ell \quad 5-2-11$$

ie. the marginal indirect utility of the i th price is diminishing with respect to increasing wealth.

j is, of course, a non-negative integer so, in reality, it is meaningless to talk of functions being continuous with respect to j . Nevertheless, for analytic convenience, it will be assumed that j is continuous and that psychic search costs, financial search costs and utility expected ex ante at the end of period T_1 are all functions which are differentiable with respect to j .

Assumption: $j \geq 0$ is continuous. 5-2-12

Assumption: $E[I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) | j, w], K_1(j)$ and $c_1(j)$ are differentiable with respect to j . 5-2-13

The optimal number of observations on p_1 for a Stigleresque searcher, obtained under assumptions (5-2-12) and (5-2-13), will be denoted by j^* . It is shown in Section 5-3 that, since I possesses properties (5-2-7) to (5-2-11), the absolute error between j^* and the actual integer optimum j^0 is always less than unity. Furthermore, the actual integer optimum j^0 can be located simply by comparing the utilities expected ex ante at the end of period T_1 for the two integer values of j within one unit either side of j^* .

It is also assumed that marginal financial search costs are positive and non-decreasing with respect to j and that marginal psychic search costs are non-decreasing with respect to j .

Assumption: $\frac{\partial c_1(j)}{\partial j} = c_1'(j) \geq 0$; $\frac{\partial^2 c_1(j)}{\partial j^2} = c_1''(j) \geq 0$ for all $j \geq 0$ 5-2-14

Assumption: $\frac{\partial K_1(j)}{\partial j^2} \geq 0$ for all $j \geq 0$ 5-2-15

Note that (5-2-15) does not require all marginal psychic search costs to be positive. It permits the searcher some psychic enjoyment from taking at least his initial observations on p_1 .

SECTION 5-3: THE STIGLER SEARCH MODEL WITHOUT FIXED-ORDER QUANTITY RESTRICTIONS

Prior to search the Stigleresque searcher can evaluate $E[I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) | j, w]$, the level of utility he expects ex ante to enjoy after taking j observations on p_1 in period T_1 . However,

he will incur psychic costs while searching through period T_1 . Prior to search, therefore, he will choose to make j^* observations on p_1 where j^* maximises the difference, with respect to j , between the utility expected at the end of period T_1 and the psychic costs incurred in attaining it. It is assumed that the consumer's indirect utility function is additively separable with respect to psychic search costs so that the consumer's ex ante expected utility, net of psychic search costs, from taking j observations on p_1 is

$$Q(j) = E[I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) | j, w] - K_1(j) \quad 5-3-1$$

The Stigleresque searcher's problem is to choose j^* such that

$$Q(j^*) = \max_{j \geq 0} Q(j) \quad 5-3-2$$

Two comments on the Stigleresque searcher's decision-making must be made at the outset. First, even if it is possible, there is no purpose to a Stigleresque searcher's updating of his prior p.d.f. on w , $g(w)$. The sequential searcher's gain from refining $g(w)$ with information about w gleaned from his observations on p_1 is that it enables him to more accurately decide upon the best length of search. The Stigleresque searcher's selection of the best length of search, j^* , is an ex ante decision which therefore utilises $g(w)$ and the constraints of the search problem are such that j^* is not revised as search proceeds. Since the Stigleresque search length is not revised, there is no gain from revision of $g(w)$. Stigleresque search will often be optimal in problems where price information does not flow to the searcher until the end of a period, even though the cost of the observations may have been incurred earlier in the period. For example, a consumer may write to j sellers of a commodity asking for price quotations. The

sellers' replies are mailed back, arriving at the end of period T_1 . The consumer's financial costs of these observations were all incurred at the beginning of the period when writing and mailing the requests for information. Additional psychic costs, perhaps anxiety or frustration, would accumulate throughout the period. No information on prices is obtained from the sellers until the end of the period so it is not possible to revise $g(w)$ within the period. Second, unlike a sequential searcher, a Stigleresque searcher will never choose to alter any part of his ex ante optimal search path $i_1^*, \dots, i_{j^*}^*$. Optimal Stigleresque search paths are examined in Section 5-4.

The Stigleresque searcher chooses to make j^* observations on p_1 where j^* satisfies the necessary maximising conditions

$$\frac{\partial Q}{\partial j} \Big|_{j=j^*} \leq 0; \quad j^* \cdot \frac{\partial Q}{\partial j} \Big|_{j=j^*} = 0; \quad j^* \geq 0 \quad 5-3-3$$

The sufficient maximising conditions are proved satisfied in Proposition 5-3-5.

$$(i) \quad \text{if } j^*=0, \quad \frac{\partial Q}{\partial j} \Big|_{j=0} \leq 0 \text{ and search does not occur} \quad 5-3-4$$

$$(ii) \quad \text{if } j^* \geq 1, \quad \frac{\partial Q}{\partial j} \Big|_{j=j^*} = 0 \text{ and at least one observation on } p_1 \text{ is taken.} \quad 5-3-5$$

$$(iii) \quad \text{if } 0 < j^* < 1, \quad \frac{\partial Q}{\partial j} \Big|_{j=j^*} = 0 \text{ and either} \quad 5-3-6$$

- (a) $Q(0) > Q(1)$ and search does not occur
or (b) $Q(0) < Q(1)$ and exactly one observation on p_1 is taken,

From (5-2-6) and (5-3-1),

$$Q(j) = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) h(p_1^{\min} | j, w) dp_1^{\min} - K_1(j) \quad 5-3-7$$

Differentiating (5-3-7) with respect to j gives

$$\begin{aligned} \frac{\partial Q}{\partial j} = & \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} + \\ & \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial j} h(p_1^{\min} | j, w) dp_1^{\min} - \frac{\partial K_1}{\partial j} \end{aligned} \quad 5-3-8$$

By (5-2-4), $\bar{M}_1(j) = M - c_1(j)$

$$\text{so } \frac{\partial \bar{M}_1}{\partial j} = -c_1'(j) \quad 5-3-9$$

Therefore,

$$\frac{\partial I}{\partial j} = \frac{\partial I}{\partial \bar{M}_1} \frac{\partial \bar{M}_1}{\partial j} = -c_1'(j) \frac{\partial I}{\partial \bar{M}_1} \quad 5-3-10$$

Combining (5-3-10) and (5-3-8) shows

$$\begin{aligned} \frac{\partial Q}{\partial j} = & \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \\ & - c_1'(j) \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial \bar{M}_1} h(p_1^{\min} | j, w) dp_1^{\min} - \frac{\partial K_1}{\partial j} \end{aligned} \quad 5-3-11$$

The first term on the R.H.S. of (5-3-11) is the marginal expected

utility of an increase in the likelihood of a lower price for

commodity 1 being revealed by the extra amount of search undertaken.

This quantity, which is dependent on the form of the p.d.f. of selling prices, is proved to be positive and diminishing with respect to j

by Proposition 5-3-3.

The second term on the R.H.S. of (5-3-11) is the marginal expected utility of a decrease of $c_1'(j)$ in the searcher's wealth caused by extending the number of observations on p_1 from j . This term is identically zero if there are no financial costs of search, $c_1 \equiv 0$, since then $c_1'(j) \equiv 0$. Proposition 5-3-4 proves assumptions (5-2-9) and (5-2-10) sufficient for this quantity to be negative and diminishing with respect to j .

The third term on the R.H.S. of (5-3-11) is the marginal psychic cost incurred by extending search from j observations on p_1 . This quantity is identically zero if there are no psychic search costs.

The following two lemmas are used to establish the behaviour of the first and second terms of the R.H.S. of (5-3-11) with respect to j . Lemma 5-3-1 establishes a result similar to that offered by Lemma 3-5-1, which considers changes in the p.d.f. of selling prices caused by changes in the value of the conditioning parameter w . In Lemma 5-3-1, however, the change is to the sample size j and not to the p.d.f. of selling prices from which samples are taken.

Lemma 5-3-1:

$$\int_{p_1^L}^{p_1^U} g(p_1^{\min}) \cdot \frac{\partial h(p_1^{\min} | j, w)}{\partial j} \cdot dp_1^{\min} > 0 \text{ for all } j > 0$$

$$\text{if } \frac{\partial g}{\partial p_1^{\min}} < 0 \text{ for all } p_1^{\min} \in [p_1^L, p_1^U],$$

Proof:

$h(p_1^{\min} | j, w)$ is a proper p.d.f. Therefore,

$$\int_{p_1^L}^{p_1^U} h(p_1^{\min} | j, w) dp_1^{\min} = 1$$

5-3-12

and

$$\frac{\partial}{\partial j} \int_{p_1^L}^{p_1^U} h(p_1^{\min}|j, w) dp_1^{\min} = \int_{p_1^L}^{p_1^U} \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} = 0 \quad 5-3-13$$

It is impossible for $\frac{\partial h(p_1^{\min}|j, w)}{\partial j} \equiv 0$, so (5-3-13) requires

$$\int_{R^+} \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} + \int_{R^-} \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} = 0 \quad 5-3-14$$

$$\text{where } R^+ = \{p_1^{\min} | \frac{\partial h(p_1^{\min}|j, w)}{\partial j} > 0\} \quad 5-3-15$$

$$\text{and } R^- = \{p_1^{\min} | \frac{\partial h(p_1^{\min}|j, w)}{\partial j} < 0\} \quad 5-3-16$$

$$\text{Let } R^0 = \{p_1^{\min} | h(p_1^{\min}|j, w) = 0\} \quad 5-3-17$$

Consider the equation $\frac{\partial h(p_1^{\min}|j, w)}{\partial j} = 0$ ie.

$$\frac{\partial h(p_1^{\min}|j, w)}{\partial j} = \frac{\partial}{\partial j} (j(1-F(p_1^{\min}|w))^{j-1} f(p_1^{\min}|w)) \quad 5-3-18$$

$$= (1-F(p_1^{\min}|w))^{j-1} f(p_1^{\min}|w) (1+j \cdot \ln(1-F(p_1^{\min}|w))) = 0 \quad 5-3-19$$

The solutions to (5-3-19) are all $p_1^{\min} \in R^0$ and $p_1^{\min} = r$ where

$$1 + j \cdot \ln(1-F(r|w)) = 0 \quad 5-3-20$$

r is unique because $F(p_1^{\min}|w)$ is a monotonic increasing function in p_1^{\min} . By definition,

$$R^+ \cap R^- = R^+ \cap R^0 = R^- \cap R^0 = R^+ \cap r = R^- \cap r = R^0 \cap r = \emptyset \quad 5-3-21$$

and

$$R^+ \cup R^- \cup R^0 \cup r = [p_1^L, p_1^U] \quad 5-3-22$$

From (5-3-20), if $p_1^{\min} < r$, $1 + j \cdot \ln(1 - F(p_1^{\min}|w)) > 0$ 5-3-23

and, if $p_1^{\min} > r$, $1 + j \cdot \ln(1 - F(p_1^{\min}|w)) < 0$ 5-3-24

Combining (5-3-19), (5-3-23) and (5-3-24) shows

$$\frac{\partial h(p_1^{\min}|j, w)}{\partial j} \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ as } p_1^{\min} \begin{matrix} < \\ > \end{matrix} r \quad 5-3-25$$

r is therefore an upper bound for R^+ and a lower bound for R^- .

Also, $\frac{\partial h(p_1^{\min}|j, w)}{\partial j} = 0$ for all $p_1^{\min} \in R^0$ 5-3-26

since $h(p_1^{\min}|j, w) = 0$ for all $p_1^{\min} \in R^0$. Combining (5-3-14), (5-3-15) (5-3-16), (5-3-21), (5-3-22), (5-3-25) and (5-3-26) shows

$$\int_{p_1^L}^r \frac{\partial h(p_1^{\min}|j, w)}{\partial j} \cdot dp_1^{\min} + \int_r^{p_1^U} \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} = 0 \quad 5-3-27$$

(a) Suppose $\frac{\partial g}{\partial p_1^{\min}} \geq 0$ for all $p_1^{\min} \in [p_1^L, p_1^U]$. Then

$$g(p_1^{\min}) \leq g(r) \quad \text{for all } p_1^L \leq p_1^{\min} \leq r \quad 5-3-28$$

and

$$g(r) \leq g(p_1^{\min}) \quad \text{for all } r \leq p_1^{\min} \leq p_1^U \quad 5-3-29$$

From (5-3-28) and (5-3-25),

$$\int_{p_1^L}^r g(p_1^{\min}) \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} \leq g(r) \int_{p_1^L}^r \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} \quad 5-3-30$$

From (5-3-29) and (5-3-25),

$$\int_r^{p_1^U} g(p_1^{\min}) \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} \leq g(r) \int_r^{p_1^U} \frac{\partial h(p_1^{\min}|j, w)}{\partial j} dp_1^{\min} \quad 5-3-31$$

Combining (5-3-30) and (5-3-31) shows

$$\int_{p_1^L}^{p_1^U} g(p_1^{\min}) \frac{\partial h(p_1^{\min}|j,w)}{\partial j} dp_1^{\min} \leq g(r) \int_{p_1^L}^{p_1^U} \frac{\partial h(p_1^{\min}|j,w)}{\partial j} dp_1^{\min} = 0 \quad 5-3-32$$

by (5-3-27).

(b) Similarly, if $\frac{\partial g}{\partial p_1^{\min}} \leq 0$ for all $p_1^{\min} \in [p_1^L, p_1^U]$, then

$$\int_{p_1^L}^{p_1^U} g(p_1^{\min}) \frac{\partial h(p_1^{\min}|j,w)}{\partial j} dp_1^{\min} \geq g(r) \int_{p_1^L}^{p_1^U} \frac{\partial h(p_1^{\min}|j,w)}{\partial j} dp_1^{\min} = 0 \quad 5-3-33$$

Q.E.D.

Lemma 5-3-2:

$$\int_{p_1^L}^{p_1^U} g(p_1^{\min}) \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} dp_1^{\min} < 0 \text{ for all } j > 0$$

if $\frac{\partial g}{\partial p_1^{\min}} < 0$ for all $p_1^{\min} \in [p_1^L, p_1^U]$.

Proof:

The proof of this lemma closely follows the arguments used in proving Lemma 5-3-1. From (5-3-13),

$$\frac{\partial^2}{\partial j^2} \int_{p_1^L}^{p_1^U} h(p_1^{\min}|j,w) dp_1^{\min} = \int_{p_1^L}^{p_1^U} \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} dp_1^{\min} = 0 \quad 5-3-34$$

$\frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} \equiv 0$ is impossible so (5-3-34) requires

$$\int_{R^{'+}} \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} dp_1^{\min} + \int_{R'^{-}} \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} dp_1^{\min} = 0 \quad 5-3-35$$

$$\text{where } R^{'+} = \{p_1^{\min} | \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} > 0\} \quad 5-3-36$$

$$\text{and } R'^{-} = \{p_1^{\min} | \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} < 0\} \quad 5-3-37$$

$$\text{Consider the equation } \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} = 0 \quad \text{ie.}$$

$$\begin{aligned} \frac{\partial^2 h(p_1^{\min}|j,w)}{\partial j^2} &= \frac{\partial^2}{\partial j^2} (j(1-F(p_1^{\min}|w))^{j-1} f(p_1^{\min}|w)) \\ &= (1-F(p_1^{\min}|w))^{j-1} f(p_1^{\min}|w) \ln(1-F(p_1^{\min}|w)) (2+j \cdot \ln(1-F(p_1^{\min}|w))) = 0 \end{aligned} \quad 5-3-38$$

The solutions to (5-3-38) are all $p_1^{\min} \in R^0$ and $p_1^{\min} = r'$ where R^0 is defined by (5-3-17) and r' is such that

$$2 + j \cdot \ln(1-F(r'|w)) = 0 \quad 5-3-39$$

r' is unique because $F(p_1^{\min}|w)$ is a monotonic increasing function in p_1^{\min} . By definition,

$$R^{'+} \cap R'^{-} = R^{'+} \cap R^0 = R'^{-} \cap R^0 = R^{'+} \cap r' = R'^{-} \cap r' = R^0 \cap r' = \emptyset \quad 5-3-40$$

$$\text{and } R^{'+} \cup R'^{-} \cup R^0 \cup r' = [p_1^L, p_1^U]. \quad 5-3-41$$

$$\text{From (5-3-39), if } p_1^{\min} < r', 2 + j \cdot \ln(1-F(p_1^{\min}|w)) > 0 \quad 5-3-42$$

$$\text{and, if } p_1^{\min} > r', 2 + j \cdot \ln(1-F(p_1^{\min}|w)) < 0 \quad 5-3-43$$

Combining (5-3-38), (5-3-42) and (5-3-43) shows

$$\frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} \begin{matrix} \leq \\ \geq \end{matrix} 0 \quad \text{as } p_1^{\min} < r' \quad 5-3-44$$

r' is therefore an upper bound for R'^{-} and a lower bound for R'^{+} .

Hence, from (5-3-44), (5-3-35), (5-3-36), (5-3-37), (5-3-40) and (5-3-41),

$$\int_{p_1^L}^{r'} \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} + \int_{r'}^{p_1^U} \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} = 0 \quad 5-3-45$$

(a) Suppose $\frac{\partial g}{\partial p_1^{\min}} \geq 0$ for all $p_1^{\min} \in [p_1^L, p_1^U]$. Then, by

(5-3-28), (5-3-29) and (5-3-44),

$$\int_{p_1^L}^{r'} g(p_1^{\min}) \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} \geq g(r') \int_{p_1^L}^{r'} \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} \quad 5-3-46$$

and

$$\int_{r'}^{p_1^U} g(p_1^{\min}) \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} \geq g(r') \int_{r'}^{p_1^U} \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} \quad 5-3-47$$

Addition of (5-3-46) and (5-3-47) shows

$$\int_{p_1^L}^{p_1^U} g(p_1^{\min}) \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} \geq g(r') \int_{p_1^L}^{p_1^U} \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} = 0 \quad 5-3-48$$

by (5-3-45).

(b) Similarly, if $\frac{\partial g}{\partial p_1^{\min}} \leq 0$ for all $p_1^{\min} \in [p_1^L, p_1^U]$, then

$$\int_{p_1^L}^{p_1^U} g(p_1^{\min}) \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} \leq g(r^*) \int_{p_1^L}^{p_1^U} \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} = 0$$

5-3-49

by (5-3-45).

Q.E.D.

Proposition 5-3-3:

$$(i) \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} > 0$$

$$(ii) \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} < 0$$

Proof:

By (5-2-7), $\frac{\partial I}{\partial p_1^{\min}} < 0$. Therefore (i) is proved by applying

the result of Lemma 3-5-1 and (ii) is proved by applying the result of Lemma 3-5-2.

Q.E.D.

Proposition 5-3-3 proves that the utility the consumer expects to enjoy at the end of period T_1 increases at a decreasing rate as the number of observations taken on p_1 increases without financial penalty.

Proposition 5-3-4:

$$(i) \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial \bar{M}_1} h(p_1^{\min} | j, w) dp_1^{\min} > 0$$

$$(ii) \int_{p_1^L}^{p_1^U} \frac{\partial^2 I}{\partial \bar{M}_1^2} h(p_1^{\min} | j, w) dp_1^{\min} < 0$$

Proof:

(i) and (ii) follow immediately from (5-2-9), that $\frac{\partial I}{\partial \bar{M}_1} > 0$,

and from (5-2-10), that $\frac{\partial^2 I}{\partial \bar{M}_1^2} < 0$.

Q.E.D.

The results of Propositions 5-3-3 and 5-3-4 can be used to show that assumptions (5-2-7), (5-2-10), (5-2-11), (5-2-14) and (5-2-15) are sufficient for $Q(j)$ to be strictly concave with respect to j . This ensures j^* is unique and that $Q(j^*)$ is a global maximum over all $Q(j)$, $j \geq 0$. Concavity of $Q(j)$ with respect to j is equivalent to diminishing marginal ex ante expected net utility with respect to j .

Proposition 5-3-5:

(i) The marginal ex ante expected net utility from search is diminishing with respect to j .

(ii) j^* is unique.

Proof:

From (5-3-11),

$$\frac{\partial^2 Q}{\partial j^2} = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} - 2c_1'(j) \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial \bar{M}_1} \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min}$$

$$+(c_1'(j))^2 \int_{p_1^L}^{p_1^U} \frac{\partial^2 I}{\partial \bar{M}_1^2} h(p_1^{\min} | j, w) dp_1^{\min} - c_1''(j) \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial \bar{M}_1} h(p_1^{\min} | j, w) dp_1^{\min} - \frac{\partial^2 K_1}{\partial j^2}$$

$\frac{\partial}{\partial p_1^{\min}} \left(\frac{\partial I}{\partial \bar{M}_1} \right) < 0$, by (5-2-11), so by Lemma 3-5-1,

$$\int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial \bar{M}_1} \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} > 0 \quad 5-3-51$$

Combining (5-3-50), (5-3-51), (ii) of Proposition 5-3-3, (ii) of Proposition 5-3-4, (5-2-14) and (5-2-15) proves (i) above, that

$\frac{\partial^2 Q}{\partial j^2} < 0$. $Q(j)$ is therefore strictly concave with respect to j and

j^* is globally optimal and unique.

Q.E.D.

In Section 1-4, search models without either financial or psychic search costs (Type 1 variations) were dismissed as unrealistic and uninteresting because optimal search always continued until the lowest price for commodity 1, p_1^L , was discovered. This result is apparent from (5-3-2) where the first of the necessary maximising conditions is

$$\int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} - c_1'(j) \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial \bar{M}_1} h(p_1^{\min} | j, w) dp_1^{\min}$$

$$\leq \frac{\partial K_1}{\partial j}$$

5-3-52

If there are neither financial nor psychic costs of search (5-3-52) becomes

$$\int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \leq 0 \quad 5-3-53$$

Since Proposition 5-3-3 shows the L.H.S. of (5-3-53) is strictly positive for finite j , (5-3-53) can only be true if an infinite number of observations are taken on p_1 ie. $j^* = \infty$. This is because, as $j \rightarrow \infty$, the p.d.f. of minima of samples of size j collapses to a degenerate p.d.f. centred on p_1^L . The marginal utility of an additional observation on p_1 , therefore, becomes vanishingly small

$$\lim_{j \rightarrow \infty} \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} = \frac{\partial I(p_1^L, p_2, \dots, p_\ell, M)}{\partial j} = 0 \quad 5-3-54$$

Either one of (5-3-55) or (5-3-56) is sufficient for $j^* < \infty$.

$$\lim_{j \rightarrow \infty} K_1(j) > I(p_1^L, p_2, \dots, p_\ell, M) \quad 5-3-55$$

$$\lim_{j \rightarrow \infty} c_1(j) > M \quad 5-3-56$$

$I(p_1^L, p_2, \dots, p_\ell, M)$ is the highest possible level of utility attainable in the absence of any search costs at all. If, as in (5-3-55), accumulated psychic search costs eventually exceed this utility level, then the optimal number of observations on p_1 must be finite.

(5-3-56) makes the obvious comment that a Stigleresque searcher will make only a finite number of observations on p_1

if continuing without limit bankrupts him.

(5-3-52) shows that $j^* = 0$ because the marginal gain in utility from one observation on p_1 is exceeded by either, or both, of the marginal changes in utility due to financial or psychic search costs. Of course, if $c \equiv 0$ and $\frac{\partial K_1}{\partial j}|_{j=1} < 0$, then $j^* \geq 1$, since $c \equiv 0$ and (i) of Proposition 5-3-3 guarantee that marginal ex ante expected net utility of the first observation on p_1 is positive. Search will always take place.

If assumptions (5-2-14) or (5-2-15) are revoked, ex ante expected net utility $Q(j)$ need no longer be strictly concave with respect to j , but j^* will still be unique except in rather unusual circumstances. Again consider the necessary maximising condition in (5-3-2) of $\frac{\partial Q}{\partial j}|_{j=j^*} = 0$ for $j^* > 0$.

From (5-3-11),

$$\int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, \bar{M}_1) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} - c_1'(j) \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial M_1} h(p_1^{\min} | j, w) dp_1^{\min} = \frac{\partial K_1}{\partial j} \quad 5-3-57$$

The slope of the L.H.S. of (5-3-57) has been proved to be strictly decreasing with respect to j by Propositions 5-3-3 and 5-3-4 (see Figures 5-3-1 and 5-3-2) so, unless $\frac{\partial K_1}{\partial j}$ is strictly equal to the L.H.S. for all j in some continuous interval, there will be a sequence of discrete solutions j_1', j_2', \dots to (5-3-57). Some of the solutions j_1', j_2', \dots will represent local minima or points

Figure 5-3-1

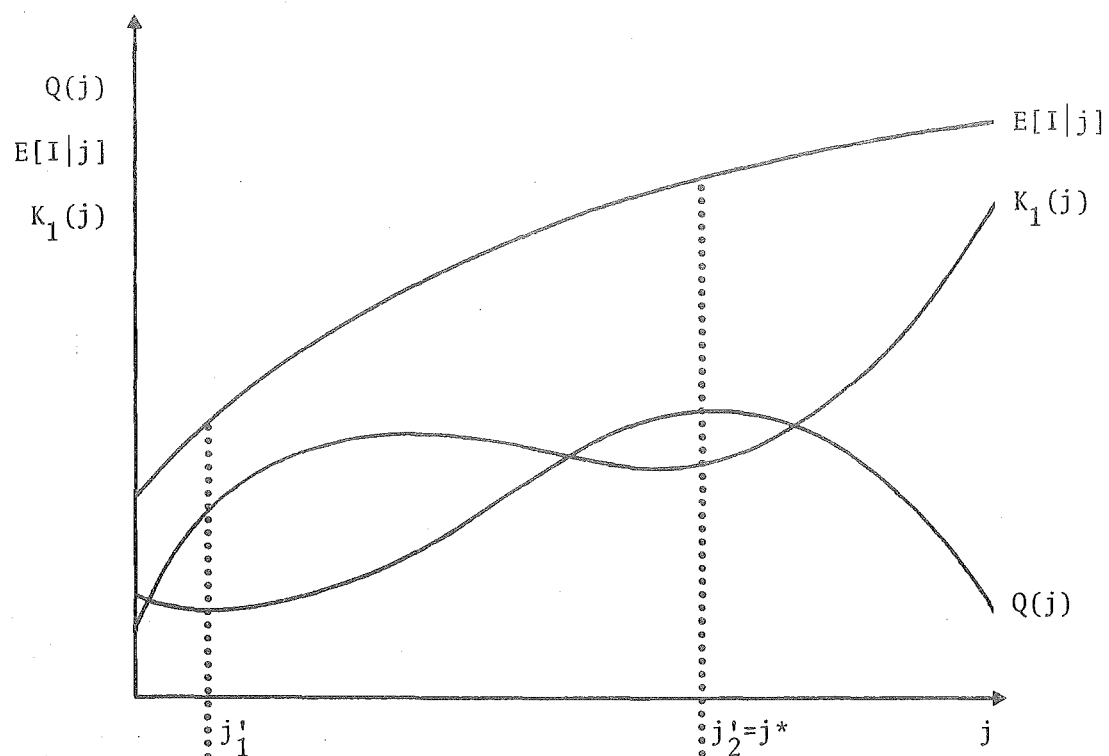
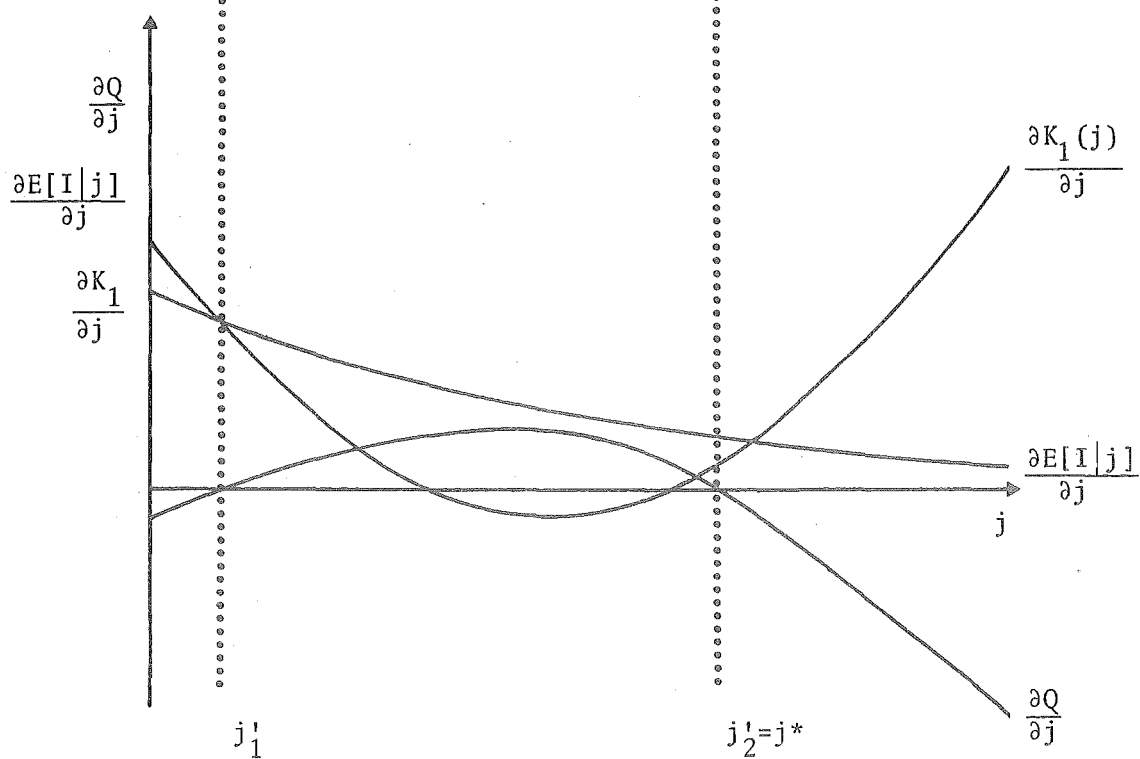


Figure 5-3-2



of inflection of $Q(j)$. The remaining solutions will represent local maxima. The Stigleresque searcher will choose as j^* the $j_i^!$ for which $Q(j)$ is greatest. j^* is the global maximum ie.

$$j^* = j_i^! \text{ where } Q(j_i^!) \geq Q(j_k^!) \text{ for all } k = 1, 2, \dots \quad 5-3-58$$

j^* will be unique unless psychic and financial costs are such that $Q(j_i^!) = Q(j_k^!)$ for some $j_i^! \neq j_k^!$. While this is possible, it is not likely for most functions I and K . In reality it is even less likely since j^* will necessarily be an integer. Should such a tie occur, however, the Stigleresque searcher is indifferent between the two sample sizes and can use some tie breaking procedure such as flipping a coin to choose one search length over the other.

Comparative statics results are obtained from the Stigleresque search model in Section 5-5. The reader is reminded of the discussion of Section 1-4 which showed the difficulty of doing this when the search model is constrained by a fixed-order quantity restriction.

The next section discusses the nature of a Stigleresque searcher's optimal search path.

SECTION 5-4: OPTIMAL SEARCH PATHS FOR STIGLERESQUE SEARCH

Chapter 3 discussed why sequential search paths should be considered and some of the factors which will influence a sequential searcher's choice of sellers to contact. There the discussion was couched in financial terms rather than utility terms. The conclusions reached were that, for a given search length, the optimal search path was the search path of least expected total expenditure on commodity 1. Under assumption (3-2-3), that the marginal financial

cost of any observation on p_1 is independent of any characteristic of any other observation on p_1 , it was shown that the optimal sequential search path was characterised by a combination of non-decreasing marginal financial costs and non-increasing marginal benefits. No such directional properties need exist for a Stigleresque searcher's optimal search path.

Definition:

The optimal Stigleresque search path is $i_1^*, \dots, i_{j^*}^*$ where

$$Q(i_1^*, \dots, i_{j^*}^*) = E[I(p_1^{\min}, p_2, \dots, p_\ell, M - \sum_{m=1}^{j^*} c_{i_m^*}) | w, i_1^*, \dots, i_{j^*}^*] - \sum_{m=1}^{j^*} K_1 i_m^* \\ \geq E[I(p_1^{\min}, p_2, \dots, p_\ell, M - \sum_{m=1}^j c_{i_m}) | w, i_1, \dots, i_j] - \sum_{m=1}^j K_1 i_m \quad 5-4-1$$

for any search path i_1, \dots, i_j .

The optimal search path is the path for which ex ante expected net utility is greatest. Some results are immediate from (5-4-1). Propositions 5-4-1 and 5-4-2 should be compared to Proposition 3-3-2.

Proposition 5-4-1:

If there are no psychic search costs and no differences in sellers' relative pricing behaviour, the optimal search path is $i_1^*, \dots, i_{j^*}^*$ where

$$\sum_{m=1}^{j^*} c_{i_m^*} \leq \sum_{m=1}^{j^*} c_{i_m}$$

for any search path i_1, \dots, i_{j^*} .

Proof:

Consider any search path i_1, \dots, i_j . Since $K \equiv 0$ the ex ante expected net utility from following this search path is

$$Q(i_1, \dots, i_j) = E[I(p_1^{\min}, p_2, \dots, p_\ell, M - \sum_{m=1}^j c_{i_m}) | w, i_1, \dots, i_j] \quad 5-4-2$$

From (5-2-9),
$$\frac{\partial I}{\partial (M - \sum_{m=1}^j c_{i_m})} > 0 \quad 5-4-3$$

Since there are no differences in sellers' relative pricing behaviour, $Q(i_1, \dots, i_j)$ is greatest if $\sum_{m=1}^j c_{i_m}$ is least. Therefore,

for any given number j of observations on p_1 , the preferred search path is i_1^*, \dots, i_j^* where $\sum_{m=1}^j c_{i_m^*} \leq \sum_{m=1}^j c_{i_m}$ for any search path

i_1, \dots, i_j .

Q.E.D.

Proposition 5-4-2:

If there are no financial search costs and no differences in sellers' relative pricing behaviour, the optimal search path is $i_1^*, \dots, i_{j^*}^*$ where

$$\sum_{m=1}^{j^*} K_{1i_m^*} \leq \sum_{m=1}^{j^*} K_{1i_m}$$

for any search path i_1, \dots, i_{j^*} .

Proof:

Consider any search path i_1, \dots, i_j . Since $c \equiv 0$ the ex ante expected net utility from following this search path is

$$Q(i_1, \dots, i_j) = E[I(p_1^{\min}, p_2, \dots, p_\ell, M) | j, w] - \sum_{m=1}^j K_{1i_m} \quad 5-4-4$$

Since there are no differences in sellers' relative pricing behaviour, $E[I(p_1^{\min}, p_2, \dots, p_\ell, M) | j, w]$ is the same for all search paths of length j .

The preferred search path of length j is, therefore, i_1^*, \dots, i_j^* where

$$\sum_{m=1}^j K_{1i_m^*} \leq \sum_{m=1}^j K_{1i_m} \text{ for any search path } i_1, \dots, i_j.$$

Q.E.D.

The import of Proposition 5-4-1 is that an optimal Stigleresque search path provides the minimum total financial search cost for j^* observations on p_1 . The value of j^* will be partly determined by this total cost. If the marginal financial search costs are independent (assumption (3-2-3)), then an optimal search path is to contact the j^* sellers who represent the j^* smallest marginal financial search costs to the searcher. In this respect the result of Proposition 5-4-1 is similar to the result of Proposition 3-3-2. There is, however, an essential difference. The Stigleresque searcher's optimal search path is a path for which the total financial cost of j^* observations is a minimum. If the marginal search costs are independent then the same total cost is incurred regardless of the order in which these same j^* marginal costs are incurred. In Proposition 3-3-2 the sequential searcher's optimal search path was characterised by non-decreasing marginal financial search costs because he had the opportunity to vary the number of observations taken in the light of past observations on p_1 . No such opportunity exists for the Stigleresque searcher who is

committed to exactly j^* observations on p_1 and whose interest, therefore, is in the total cost of j^* observations rather than in the manner in which the marginal costs are incurred. The Stigleresque searcher will be indifferent between all of the j^* possible search paths over the j^* sellers offering the j^* smallest marginal financial search costs.

If the marginal financial search costs are not independent, an optimal Stigleresque search path is the solution to a dynamic programming problem. Furthermore, the optimal path may be unique since, in general, interdependence of marginal costs will ensure that variations in the order in which the j^* sellers are contacted will vary the total financial search cost. The order for an optimal path will be determined as part of the solution to the dynamic programming problem. In general, this order need not be the same as the order in which marginal financial search costs are non-decreasing. Indeed, such an order may not be possible. Also, because of the interdependence of marginal costs, an optimal search path need not be a search path over the sellers representing the j^* smallest marginal financial search costs to the searcher (where marginal is with respect to the searcher's state at the start of his search).

Similar arguments can be put forward for the psychic costs determined optimal search path described by Proposition 5-4-2.

An optimal Stigleresque search path where both financial and psychic search costs are incurred is more difficult to describe. Even if marginal financial search costs are independent and marginal

psychic search costs are independent, there will always be a strong dependence between the marginal financial and marginal psychic costs of contacting a particular seller. In general, an optimal search path for this case appears to be determinate only as the solution to a dynamic programming problem. However, if marginal financial and psychic search costs are perfectly positively correlated (ie. the seller representing the i th smallest marginal financial cost to the search also represents the i th smallest marginal psychic cost to the searcher, for all $i \geq 1$), then the optimal Stigleresque search path can again be characterised by increasing marginal costs. This is the result of Proposition 5-4-3. It is often the case that marginal financial and psychic costs are positively correlated. For instance, the higher financial cost incurred in travelling to a more distant seller would usually be accompanied by incurring a higher psychic cost from loss of time used in travelling, the additional discomforts of travelling a longer distance and so on.

Proposition 5-4-3:

If marginal financial search costs c_{1i_m} and marginal psychic search costs K_{1i_m} are ranked identically with respect to i_m , then the optimal search path is $i_1^*, \dots, i_{j^*}^*$ where

$$\sum_{m=1}^{j^*} c_{1i_m^*} \leq \sum_{m=1}^{j^*} c_{1i_m} \quad \text{and} \quad \sum_{m=1}^{j^*} K_{1i_m^*} \leq \sum_{m=1}^{j^*} K_{1i_m}$$

for any search path i_1, \dots, i_j ,

Proof:

The ex ante expected net utility from following any search path i_1, \dots, i_j is

$$Q(i_1, \dots, i_j) = E[I(p_1^{\min}, p_2, \dots, p_\ell, M - \sum_{m=1}^j c_{1i_m}) | w, i_1, \dots, i_j] \leq \sum_{m=1}^j K_{1i_m} \quad 5-4-5$$

Let i_1^*, \dots, i_j^* be the search path of length j for which

$$\sum_{m=1}^j c_{1i_m^*} \leq \sum_{m=1}^j c_{1i_m} \quad \text{and} \quad \sum_{m=1}^j K_{1i_m^*} \leq \sum_{m=1}^j K_{1i_m} \quad 5-4-6$$

for any search path i_1, \dots, i_j . There are no differences in sellers' relative pricing behaviour so

$$\begin{aligned} & E[I(p_1^{\min}, p_2, \dots, p_\ell, M - \sum_{m=1}^j c_{1i_m^*}) | w, i_1^*, \dots, i_j^*] \\ & \geq E[I(p_1^{\min}, p_2, \dots, p_\ell, M - \sum_{m=1}^j c_{1i_m}) | w, i_1, \dots, i_j] \end{aligned} \quad 5-4-7$$

since $\frac{\partial I}{\partial (M - \sum_{m=1}^j c_{1i_m})} > 0$ by (5-2-9). Combining (5-4-5), (5-4-6)

$$\text{and (5-4-7) shows } Q(i_1^*, \dots, i_j^*) \geq Q(i_1, \dots, i_j) \quad 5-4-8$$

i_1^*, \dots, i_j^* , satisfying (5-4-6), is therefore the preferred search path of length j .

Q.E.D.

The next section provides some comparative statics results for the Stigleresque search model with variable demand.

SECTION 5-5: THE DEPENDENCE OF THE QUANTITY OF SEARCH UPON WEALTH, PSYCHIC SEARCH COSTS AND THE NATURE OF THE SEARCHED- FOR COMMODITY

Section 1-4 showed comparative statics results could be obtained in search models with fixed demand restrictions, but only if qualified

by a requirement that changes in the model's parameters did not exceed amounts specified by the degree of non-smoothness necessarily possessed by the indifference hypersurfaces admissible for these models. This section shows that no such qualifications are necessary when the search model is not constrained by a fixed demand restriction. The qualification examined in Section 1-4 was particularly serious when financial search costs were present in the model. Financial search costs change the searcher's disposable wealth and can cause a contradiction with the fixed demand restriction if they become sufficiently great. Section 1-4 examined the restrictions a fixed demand restriction $x_1^* = \hat{x}_1$ implies for the searcher's utility function. Sections 2-1 and 2-7 point out that, even if the searcher's utility function satisfies these restrictions, an assumption that $x_1^* = \hat{x}_1$ is always feasible is still necessary to avoid a contradiction with the fixed demand restriction $x_1^* = \hat{x}_1$.

The approach adopted in the search model examined in this chapter is subject to none of these qualifications. Variable demand for commodity 1 allows both financial and psychic costs of search to be included in a neo-classical analysis of consumer search models in a quite natural manner. While the present search model is Stigleresque, the reader is reminded that the same approach can be adopted for a sequential search model.

Comparative statics results are presented first for changes in p_2, \dots, p_ℓ , second for changes in M , and third for changes in marginal psychic search costs.

Proposition 5-5-1:

If $c \equiv 0$, and if $\lim_{j \rightarrow \infty} K_1(j) > I(p_1^L, p_2, \dots, p_\ell, M)$, then

$$\frac{\partial j^*}{\partial p_i} < 0 \quad \text{as} \quad \frac{\partial^2 I}{\partial p_i \partial p_1^{\min}} > 0.$$

Proof:

If $c \equiv 0$, ex ante expected net utility at the end of period T_1 is

$$Q(j) = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) h(p_1^{\min} | j, w) dp_1^{\min} - K_1(j) = EI(j) - K_1(j) \quad 5-5-1$$

where $EI(j)$ is the ex ante expected utility exclusive of psychic search costs. Proposition 5-3-3 established that $EI(j)$ is increasing at a decreasing rate with respect to j .

$$\begin{aligned} \text{ie. } \frac{\partial EI}{\partial j} &= \frac{\partial}{\partial j} \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) h(p_1^{\min} | j, w) dp_1^{\min} \\ &= \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} > 0 \end{aligned} \quad 5-5-2$$

$$\begin{aligned} \frac{\partial^2 EI}{\partial j^2} &= \frac{\partial}{\partial j} \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \\ &= \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial^2 h(p_1^{\min} | j, w)}{\partial j^2} dp_1^{\min} < 0 \end{aligned} \quad 5-5-3$$

The optimal number of observations, j^* , is the solution to (5-3-3).

Since $\frac{\partial EI}{\partial j} > 0$, the second-order maximisation conditions require

$$\frac{\partial EI}{\partial j} = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} > \frac{\partial K_1(j)}{\partial j} \text{ as } j < j^* \quad 5-5-4$$

$\frac{\partial K_1}{\partial j}$ is unaffected by a change in p_i so the change in j^* is dependent on the change in $\frac{\partial EI}{\partial j}$ caused by the change in p_i . From (5-5-2),

$$\begin{aligned} \frac{\partial}{\partial p_i} \left(\frac{\partial EI}{\partial j} \right) &= \frac{\partial}{\partial p_i} \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \\ &= \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial p_i} \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} < 0 \text{ as } \frac{\partial^2 I}{\partial p_i \partial p_1^{\min}} < 0 \end{aligned} \quad 5-5-5$$

(a) Suppose $\frac{\partial^2 I}{\partial p_i \partial p_1^{\min}} > 0$. From (5-5-5) and (5-5-4), if p_i is

increased,

$$\begin{aligned} \frac{\partial EI(j)}{\partial j} \Big|_{j=j^*} &= \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \Big|_{j=j^*} \\ &< \frac{\partial K(j)}{\partial j} \Big|_{j=j^*} \end{aligned} \quad 5-5-6$$

Therefore, the new optimum $j' < j^*$. Hence $\frac{\partial j^*}{\partial p_i} < 0$.

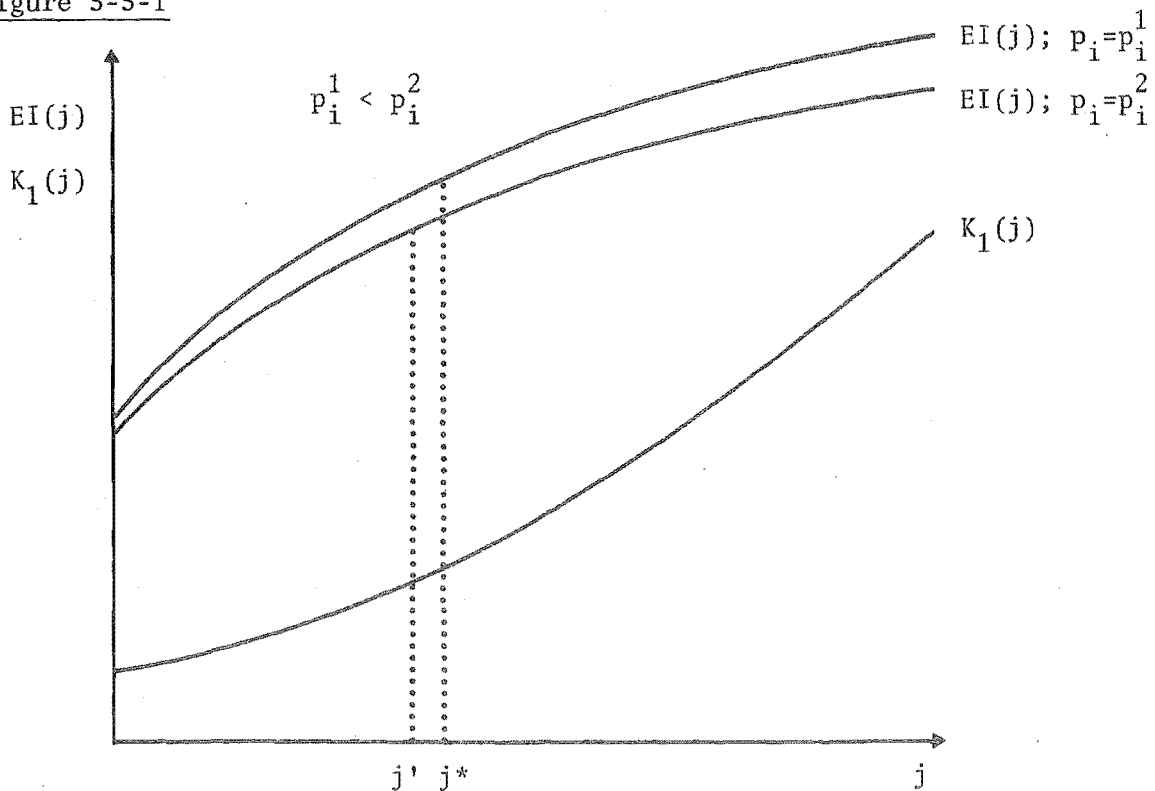
(b) Similarly, if $\frac{\partial^2 I}{\partial p_i \partial p_1} < 0$, then $\frac{\partial j^*}{\partial p_i} > 0$,

(c) Similarly, if $\frac{\partial^2 I}{\partial p_i \partial p_1} = 0$, then $\frac{\partial j^*}{\partial p_i} = 0$,

Q.E.D.

Figure 5-5-1 illustrates the results of Proposition 5-5-1 for the case $\frac{\partial^2 I}{\partial p_i \partial p_1} > 0$.

Figure 5-5-1



The conclusions of Proposition 5-5-1 are intuitive once it is realised that the signs of $\frac{\partial^2 I}{\partial p_i \partial p_1^{\min}}$ and $\frac{\partial^2 I}{\partial p_i \partial M}$ determine whether or not commodity i is a substitute or a complement for commodity 1. This can be shown by examining the first-order optimality conditions for the constrained indirect utility consumer problem (see Hadar [13, p. 201]),

$$x_1 = - \frac{\partial I / \partial M}{\partial p_1^{\min}} \quad 5-5-7$$

Comparative statics results are obtained more easily if it is assumed, like Stigler, that the marginal utility of income is constant.

$$\text{Assumption: } \frac{\partial I}{\partial M_1} = \text{constant} \quad 5-5-8$$

Then, from (5-5-7) and (5-5-8),

$$\frac{\partial x_1}{\partial p_i} = - \frac{\partial^2 I}{\partial p_1^{\min} \partial p_i} / \frac{\partial I}{\partial M_1} \quad 5-5-9$$

$\frac{\partial I}{\partial M_1} > 0$ by (5-2-9) so

$$\frac{\partial x_1}{\partial p_i} > 0 \text{ as } \frac{\partial^2 I}{\partial p_1^{\min} \partial p_i} < 0 \quad 5-5-10$$

If $\frac{\partial^2 I}{\partial p_1^{\min} \partial p_i} > 0$, then demand for commodity 1 is reduced by an

increase in p_i , implying commodity i is a gross complement for commodity 1. Since the gains from locating a lower price for

commodity 1 are thereby reduced, the incentive for additional search is also reduced.

Consequently, j^* is reduced by an increase in p_i .

If $\frac{\partial^2 I}{\partial p_1 \min \partial p_i} < 0$, then demand for commodity 1 is increased by

an increase in p_i , implying commodity i is a gross substitute for commodity 1. The incentive for locating a lower price for commodity 1 is therefore increased by an increase in p_i and j^* is thereby also increased.

If $\frac{\partial^2 I}{\partial p_1 \min \partial p_i} = 0$, then demand for commodity 1 is unaffected

by an increase in p_i so that the gains from search are not altered. Consequently, j^* is unaffected by a change in p_i .

These results are an alternative explanation of the results of Proposition 5-5-1, but only for the special case of a constant marginal utility of income.

The effect on j^* of a change in the consumer's initial wealth M is now considered.

Proposition 5-5-2:

If $c \equiv 0$ and if $\lim_{j \rightarrow \infty} K_1(j) > I(p_1^L, p_2, \dots, p_\ell, M)$, then

$$\frac{\partial j^*}{\partial M} < 0 \quad \text{as} \quad \frac{\partial^2 I}{\partial M \partial p_1 \min} > 0,$$

Proof:

The proof of this proposition is similar to the proof of Proposition 5-5-1. In Proposition 5-5-1, (5-5-2) and (5-5-3) established that $\frac{\partial EI(j)}{\partial j} > 0$ and that $\frac{\partial^2 EI(j)}{\partial j^2} < 0$, (5-5-4) gave

the second-order maximisation conditions as

$$\frac{\partial EI(j)}{\partial j} = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} > \frac{\partial K_1(j)}{\partial j}$$

$$\text{as } j < j^*$$

5-5-11

$\frac{\partial K_1}{\partial j}$ is unaffected by a change in M so the change in j^* caused by a change in M is dependent on the change in $\frac{\partial EI}{\partial j}$ caused by the change in M .

From (5-5-11),

$$\frac{\partial}{\partial M} \left(\frac{\partial EI}{\partial j} \right) = \frac{\partial}{\partial M} \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min}$$

$$= \int_{p_1^L}^{p_1^U} \frac{\partial I}{\partial M} \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} < 0 \text{ as } \frac{\partial^2 I}{\partial M \partial p_1^{\min}} < 0$$

5-5-12

(a) Suppose $\frac{\partial^2 I}{\partial M \partial p_1^{\min}} > 0$. From (5-5-12) and (5-5-11), if M is increased,

$$\frac{\partial EI(j)}{\partial j} \Big|_{j=j^*} = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \Big|_{j=j^*}$$

$$< \frac{\partial K_1(j)}{\partial j} \Big|_{j=j^*}$$

5-5-13

Therefore, the new optimum $j' < j^*$. Hence $\frac{\partial j^*}{\partial M} < 0$.

(b) Similarly, if $\frac{\partial^2 I}{\partial M \partial p_1 \min} < 0$, then $\frac{\partial j^*}{\partial M} > 0$.

(c) Similarly, if $\frac{\partial^2 I}{\partial M \partial p_1 \min} = 0$, then $\frac{\partial j^*}{\partial M} = 0$.

Q.E.D.

As with Proposition 5-5-1, the conclusions of Proposition 5-5-2 are intuitive once it is realised that the sign of $\frac{\partial^2 I}{\partial M \partial p_1 \min}$ influences whether or not commodity 1 is a superior or inferior good.

Referring to (5-5-7),

$$\frac{\partial x_1}{\partial M} = -\left(\frac{\partial I}{\partial M} \cdot \frac{\partial^2 I}{\partial M \partial p_1 \min} - \frac{\partial^2 I}{\partial M^2} \cdot \frac{\partial I}{\partial p_1 \min}\right) / \left(\frac{\partial I}{\partial M}\right)^2 \quad 5-5-14$$

$\frac{\partial I}{\partial M} > 0$ and $\frac{\partial^2 I}{\partial M^2} < 0$ by (5-2-9) and (5-2-10) and $\frac{\partial I}{\partial p_1 \min} < 0$ by

(5-2-7) so

$$\frac{\partial x_1}{\partial M} > 0 \text{ if } \frac{\partial^2 I}{\partial M \partial p_1 \min} \leq 0 \quad 5-5-15$$

$$\frac{\partial x_1}{\partial M} \leq 0 \text{ if } \frac{\partial^2 I}{\partial M \partial p_1 \min} \geq \frac{\partial^2 I}{\partial M^2} \cdot \frac{\partial I}{\partial p_1 \min} / \frac{\partial I}{\partial M} > 0 \quad 5-5-16$$

ie. commodity 1 is a superior good if $\frac{\partial^2 I}{\partial M \partial p_1 \min} \leq 0$ and a neutral or

inferior good if $\frac{\partial^2 I}{\partial M \partial p_1 \min}$ is sufficiently positive. If commodity 1

is a superior good, then an increase in wealth M will increase the demand for commodity 1 and so increase the gains from search.

Offsetting this is the diminution of the consumer's marginal utility of income due to his increased wealth. If $\frac{\partial^2 I}{\partial M \partial p_1} < 0$,

then commodity 1 is a sufficiently superior good for the increase in the gains from search to exceed the increase in the consumer's disincentive for search caused by increased wealth diminishing his marginal utility of income. Consequently, j^* is increased by an increase in wealth. If $\frac{\partial^2 I}{\partial M \partial p_1} = 0$, then, while still a

superior good, commodity 1 is only sufficiently superior for the increase in the gains from search to be exactly offset by the increase in the consumer's disincentive for search. j^* is

therefore invariant with respect to changes in consumer wealth M .

If $\frac{\partial^2 I}{\partial M \partial p_1} > 0$, the increase in the consumer's disincentive for

search exceeds the increase in the gains from search and j^* is

decreased by an increase in wealth. If $\frac{\partial^2 I}{\partial M \partial p_1}$ is sufficiently

positive (see (5-5-16)) for $\frac{\partial x_1}{\partial M} < 0$, then commodity 1 is an inferior

good. An increase in wealth M therefore not only increases the

consumer's lethargy towards search but also reduces the gains from

search. j^* will therefore decrease more rapidly with respect to

an increase in M if commodity 1 is an inferior rather than a neutral or superior good.

The effect on j^* of an increase in marginal psychic search costs is now considered. To allow such changes to be expressed formally, psychic search costs are regarded as being conditioned

by a parameter γ . To denote this the psychic cost function is written as $K(j|\gamma)$. The interpretation of γ is left open. γ may represent many factors. For example, γ may be used to denote exogenous factors such as the weather. If search is conducted on a face-to-face basis, psychic costs will be higher if it is raining than if it is a bright warm sunny day. For analytic simplicity it is assumed that marginal changes in K with respect to γ are either only monotonic increasing with respect to j or else only monotonic decreasing with respect to j . In the context of the above example, this is equivalent to saying that, if it begins to rain, the additional marginal discomfort of the fourth observation on p_1 is greater than the additional marginal discomfort of the third observation and so on.

Proposition 5-5-3:

If $c \equiv 0$ and if $\lim_{j \rightarrow \infty} K_1(j|\gamma) > I(p_1^L, p_2, \dots, p_\ell, M)$, then $\frac{\partial j^*}{\partial \gamma} > 0$ as $\frac{\partial^2 K_1(j|\gamma)}{\partial \gamma \partial j} < 0$ for all $j \geq 0$.

Proof:

In Propositions 5-5-1 and 5-5-2 the second-order maximisation conditions for an optimum of j^* observations were given as

$$\frac{\partial EI(j)}{\partial j} = \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} > \frac{\partial K_1(j|\gamma)}{\partial j}$$

as $j < j^*$ 5-5-17

$\frac{\partial EI(j)}{\partial j}$ is unaffected by a change in γ , so the change in j^* caused

by a change in γ is dependent on the change in $\frac{\partial K_1(j|\gamma)}{\partial j}$ caused by the change in γ .

(a) Suppose $\frac{\partial^2 K_1(j|\gamma)}{\partial \gamma \partial j} > 0$. From (5-5-17), if γ is increased,

$$\begin{aligned} \frac{\partial EI(j)}{\partial j} \Big|_{j=j^*} &= \int_{p_1^L}^{p_1^U} I(p_1^{\min}, p_2, \dots, p_\ell, M) \frac{\partial h(p_1^{\min} | j, w)}{\partial j} dp_1^{\min} \Big|_{j=j^*} \\ &< \frac{\partial K_1(j|\gamma)}{\partial j} \Big|_{j=j^*} \end{aligned} \quad 5-5-18$$

Therefore, the new optimum $j' < j^*$. Hence $\frac{\partial j^*}{\partial \gamma} < 0$.

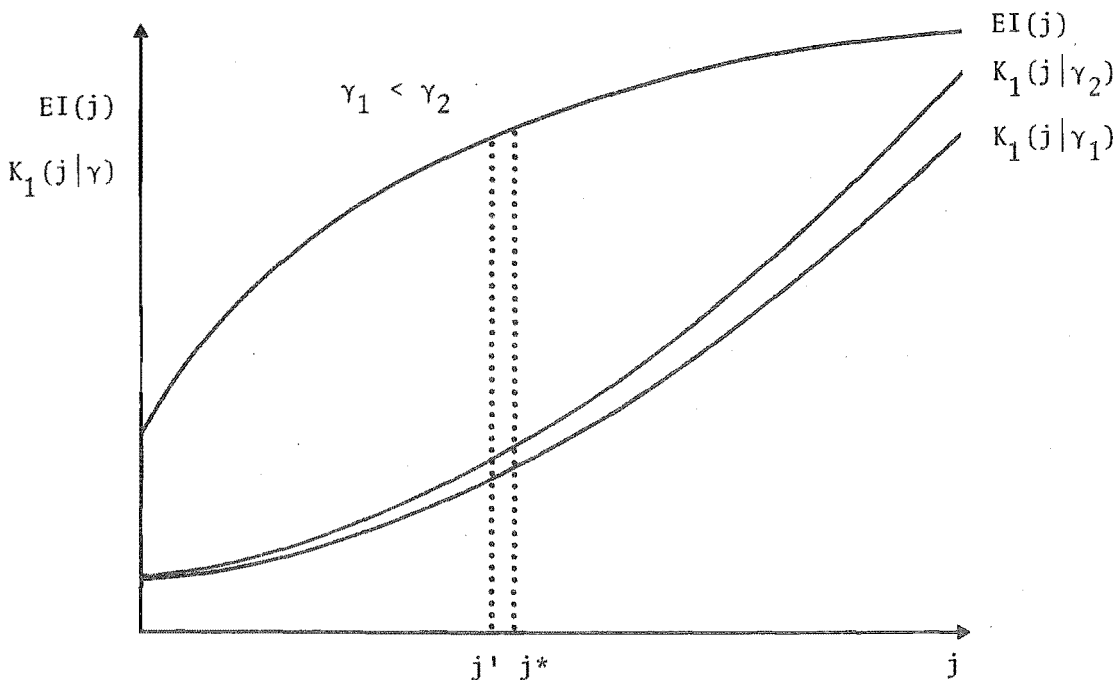
(b) Similarly, if $\frac{\partial^2 K_1(j|\gamma)}{\partial \gamma \partial j} < 0$, then $\frac{\partial j^*}{\partial \gamma} > 0$. 5-5-19

(c) Similarly, if $\frac{\partial^2 K_1(j|\gamma)}{\partial \gamma \partial j} = 0$, then $\frac{\partial j^*}{\partial \gamma} = 0$. 5-5-20

Q.E.D.

Figure 5-5-2 illustrates the results of Proposition 5-5-3 for the case of $\frac{\partial^2 K_1(j|\gamma)}{\partial \gamma \partial j} > 0$.

Figure 5-5-2



The results of Proposition 5-5-3 are intuitively obvious, If all marginal psychic search costs are increased (decreased), while marginal expected gains are not altered, then it will be optimal to decrease (increase) the number of observations taken on p_1 .

The expected minimum value of j^* observations taken on p_1 , $E[p_1^{\min}|j^*]$, is clearly a function of j^* . Variations in parameters affecting j^* will, therefore, cause variations in the searcher's expectations of the minimum observed price. The following Corollary shows how $E[p_1^{\min}|j^*]$ is affected by changes in p_2, \dots, p_ℓ , M and γ .

Corollary 5-5-4:

If $c \equiv 0$ and if $\lim_{j \rightarrow \infty} K(j|\gamma) > I(p_1^L, p_2, \dots, p_\ell, M)$, then

$$(i) \quad \frac{\partial E[p_1^{\min}|j^*]}{\partial p_i} \gtrless 0 \text{ as } \frac{\partial^2 I}{\partial p_i \partial p_1^{\min}} \gtrless 0$$

$$(ii) \quad \frac{\partial E[p_1^{\min}|j^*]}{\partial M} \gtrless 0 \text{ as } \frac{\partial^2 I}{\partial M \partial p_1^{\min}} \gtrless 0$$

$$(iii) \quad \frac{\partial E[p_1^{\min}|j^*]}{\partial \gamma} \gtrless 0 \text{ as } \frac{\partial^2 K}{\partial \gamma \partial j} \gtrless 0$$

Proof:

$$E[p_1^{\min}|j^*] = \int_{p_1^L}^{p_1^U} p_1^{\min} h(p_1^{\min}|j^*, w) dp_1^{\min}$$

$$\text{Therefore, } \frac{\partial E[p_1^{\min}|j^*]}{\partial j^*} = \int_{p_1^L}^{p_1^U} p_1^{\min} \frac{\partial h(p_1^{\min}|j^*, w)}{\partial j^*} dp_1^{\min}$$

$$\text{Also, } \frac{\partial}{\partial p_1} (p_1^{\min}) = 1 > 0$$

5-5-22

Combining (5-5-21), (5-5-22) and the result of Lemma 5-3-1 shows

$$\frac{\partial E[p_1^{\min} | j^*]}{\partial j^*} < 0$$

5-5-23

Combining (5-5-23) and the results of Proposition 5-5-1 proves

(i) above.

Combining (5-5-23) and the results of Proposition 5-5-2 proves

(ii) above.

Combining (5-5-23) and the results of Proposition 5-5-3 proves

(iii) above.

Q.E.D.

This thesis has, to this point, treated sequential and Stigleresque search behaviours separately. Sequential search was examined in Chapters 2, 3 and 4 wherein the sequential search models of Rothschild, Telser, Axell and Gastwirth were extended considerably. This chapter has made considerable extensions to the Stigleresque search model. The advantages of analysing either kind of search behaviour in utility rather than financial terms have been demonstrated and a much larger collection of comparative statics results have been assembled than was possible in the theoretical frameworks of these authors. In Section 1-4 it was promised that both Stigleresque and sequential search behaviours would be demonstrated to be opposite extremes of a more general form of adaptive search behaviour, called "generalised search". The promised demonstration is part of the development and analysis of the generalised search model presented in Chapter 6.

CHAPTER VI

GENERALISED SEARCH

SECTION 6-1: MOTIVATION OF GENERALISED SEARCH THEORY

The incomplete fashion in which the literature on search problems specifies the ways in which time constrains search is described in Section 1-3 as one of the two greatest deficiencies in the search literature. In that section, three examples were provided which demonstrated that time constraints had an important influence in determining an individual's optimal search behaviour. The last of these examples was one in which the searcher could choose to take any number of observations on p_1 in any period and could extend his search over many periods. This type of adaptive search behaviour was termed "generalised search". It was claimed that Stigleresque and sequential search behaviours are special cases of this more general behaviour. If the constraints of the problem faced by a searcher are such that, at the end of period T_1 , the expected marginal net gains from continuing search are always negative, then the generalised search rule takes on the characteristics of the Stigleresque search rule. If the constraints of the problem are such that, in any period, the expected marginal net gains from more than one observation on p_1 are always negative, then the generalised search rule takes on the characteristics of the sequential search rule. Generalised search behaviour emerges from these two of its extremes as these constraints are relaxed.

It is easy to visualise how a change in what one might call the "technology of search" can cause this emergence. Modern society has developed a variety of aids to communication and search.

Telephones, television, radio, mail services, advertising and motor vehicles have all endowed the modern searcher with the ability to make observations on prices far more rapidly and at a far smaller cost than his ancestors. Rising technological levels thus relax the constraints faced by a searcher and can cause his search behaviour to be described completely by neither Stigleresque nor sequential search patterns.

The notation required for the description of the generalised search problem was presented in Sections 1-4 and 2-2 and is listed alphabetically in Appendix 1 for the reader's convenience, along with all the notation used in all other sections of the thesis. A résumé of the generalised search notation is given here as part of the motivation of generalised search and to assist the reader in his comprehension of later sections of this chapter.

Unlike sequential or Stigleresque search, a generalised search may last any number of periods and any number of observations may be taken in any of these periods. The generalised search rule describing this behaviour is written in Section 1-4 as an ordered triple $\rho = (\xi, v, \delta)$ where ξ is a stopping rule, v is an observation number rule and δ is a terminal decision procedure.

If using search rule ρ results in a search persisting over periods T_1, \dots, T_j and if the numbers of observations on p_1 made in these periods are n_1, \dots, n_j , the searcher's terminal action space is

$$A_j = \{a_0, a_{11}, \dots, a_{1n_1}, \dots, a_{j1}, \dots, a_{jn_j}\} \quad 6-1-1$$

Terminal action a_0 is to purchase none of commodity 1. Terminal action a_{ik} is to purchase a positive quantity of commodity 1 from the k th

seller contacted in the i th period. Given the vector of price observations $y_j = (p_1^{11}, \dots, p_1^{jn_j})$, defined by (1-4-14), δ selects an element from A_j , $\delta(y_j) = a_{ik} \in A_j$. In general, therefore,

$$\delta: [p_1^L, p_1^U] \times \dots \times [p_1^L, p_1^U] \rightarrow A \quad 6-1-2$$

Note that if $j = 0$, the terminal action space is $A_0 = \{a_0\}$ ie. a non-zero quantity of commodity 1 can be demanded only if at least one seller is contacted for a price quotation. Hence,

$$\delta(y_0) \equiv a_0 \quad \text{for any terminal decision procedure } \delta. \quad 6-1-3$$

The number of observations taken in period T_{j+1} is n_{j+1} where n_{j+1} is a non-negative integer. The value n_{j+1} is determined at the beginning of period T_{j+1} , is not altered throughout period T_{j+1} and depends upon the values of prices observed in periods T_1, \dots, T_j and upon the terminal action selected by the terminal decision procedure δ . For instance, if $\delta(y_j) \equiv a_0$ for all $j \geq 0$, there is nothing to be gained by selecting a value $n_{j+1} > 0$ since none of commodity 1 will ever be purchased. Given the vector of price observations y_j and the terminal decision procedure δ , the observation number rule v determines $n_{j+1} \in I^+$, the set of non-negative integers. In general, therefore,

$$v: [p_1^L, p_1^U] \times [p_1^L, p_1^U] \times \dots \rightarrow I^+ \times I^+ \times \dots \quad 6-1-4$$

The j th component of the stopping rule ξ is ξ_j . ξ_j determines whether or not search is carried on past the end of period T_j . If $\xi_j(y_j) = S_j = 0$, search is carried on at least to the end of period T_{j+1} . If $\xi_j(y_j) = S_j = 1$, search is halted at the end of period T_j .

Once stopped, search is never recommenced. It is, however, possible that $S_j = 0$ and $n_{j+1} = 0$ so that, although search carries on through period T_{j+1} , no observations on p_1 are taken in period T_{j+1} . Such an action is rational only if more observations are taken on p_1 in a later period; ie. some $n_r > 0$ where $r > j+1$ and where $S_{j+1} = \dots = S_r = 0$. Calling a temporary halt to the taking of observations on p_1 is not an uncommon real life search phenomenon. For some reason search costs may be very high in period T_{j+1} compared to costs in period T_{j+2} . Temporary halts are a phenomenon which cannot be encompassed by either of the narrower Stigleresque or sequential search frameworks. A Stigleresque search, by definition, lasts only to the end of period T_1 . A sequential search requires that exactly one observation on p_1 is taken in every period over which search persists. A sequential searcher thus does not have the freedom of not taking his single observation in some period T_i and then taking single observations in periods subsequent to period T_i . If search is halted at the end of period T_j , no observations are taken on p_1 in any subsequent period. Hence, $S_j = 1$ implies $n_r = 0$ for all $r \geq j+1$. The converse is also true. Given a vector of price observations y_j , an observation number rule v and a terminal decision procedure δ , the stopping rule ξ determines $S_j \in \{0,1\}$ ie. $\xi_j(y_j) \in \{0,1\}$. In general, therefore,

$$\xi: [p_1^L, p_1^U] \times [p_1^L, p_1^U] \times \dots \rightarrow \{0,1\} \times \{0,1\} \times \dots \quad 6-1-5$$

The optimal generalised search rule is an adaptive rule which is used by the searcher in the following manner. At the beginning of the first period T_1 the searcher assesses the present values of all the net utilities from search expected at the end of all periods in which

it is possible for him to search. These net utilities clearly depend upon the searcher's optimal terminal decision procedure δ^* . For a given sequence n_1, n_2, \dots of numbers of observations on p_1 in periods T_1, T_2, \dots the searcher is able to decide at the end of which period he expects the highest present valued net utility. If this maximum is now, the searcher will not search and so $S_0^* = 1$. If this maximum is expected in some subsequent period, the searcher will search through at least the first period T_1 so $S_0^* = 0$. The magnitude and expected position of the maximum expected net utility depends upon the values of n_1, n_2, \dots . There will be a sequence n_1^*, n_2^*, \dots for which the magnitude of the highest present valued expected net utility is greatest. The sequence n_1^*, n_2^*, \dots and the corresponding sequence S_0^*, S_1^*, \dots are the values given ex ante by the searcher to the elements of his optimal stopping rule ξ^* and his optimal observation number rule v^* . Only the first elements of ξ^* and v^* are binding on the searcher. If $S_0^* = 0$ and $n_1^* = m$, search proceeds through period T_1 , and exactly m observations are taken on p_1 . At the end of period T_1 , the searcher will reassess the values S_1^*, S_2^*, \dots and n_2^*, n_3^*, \dots in the light of the values of his m observations $p_1^{11}, \dots, p_1^{1m}$ on p_1 . The same decision process undertaken at the start of period T_1 is now undertaken again at the start of period T_2 . Such adaptive behaviour is a direct consequence of applying Bellman's principle of optimality.

The marginal financial search cost of taking the k th observation in the i th period is c_{ik} . The number of observations taken in the i th period is $n_i \geq 0$. Therefore, by the end of period T_j , the accumulated financial cost of search is $c(j)$, defined by (1-4-16) as

$$c(j) = \sum_{i=1}^j \sum_{k=1}^{n_i} c_{ik} \quad 6-1-6$$

The searcher's net wealth at this point of his search is

$\bar{M}_j = M - c(j)$. If search stops at the end of period T_j and if the terminal decision procedure δ selects action a_0 , the searcher will allocate his net wealth of $\bar{M}_j = M - c(j)$ to purchases of commodities 2 to ℓ . If δ selects action $a_{ik} \neq a_0$ then the searcher has chosen to purchase a positive quantity of commodity 1 and, in so doing, may have chosen to incur a transaction cost t_{ik} . The searcher will be left with a net wealth of only $(\bar{M}_j - t_{ik})$ to allocate to purchases of commodities 1 to ℓ .

The psychic search cost of taking n_i observations in the i th period is $K(i, n_i)$.

Given a discount rate α , the present value of the searcher's ex ante expected net utility from using a generalised search rule $\rho = (\xi, v, \delta)$ is denoted by $V_0(\xi, v, \delta)$.

Definition:

$V_0(\xi, v, \delta)$ is the present valued ex ante expected net utility from search conducted according to the generalised search rule $\rho = (\xi, v, \delta)$.

(1-4-22) defines the optimal search rule $\rho^* = (\xi^*, v^*, \delta^*)$ as the search rule which maximises the present valued ex ante expected net utility from search. Hence, ρ^* must have the property that

$$V_0(\xi^*, v^*, \delta^*) \geq V_0(\xi, v, \delta) \quad 6-1-7$$

for any search rule $\rho = (\xi, v, \delta)$.

Definition:

$V_j(\xi, v, \delta)$ is the present value, at the end of period T_j , of the expected net utility from search conducted according to the

search rule $\rho = (\xi, v, \delta)$.

$V_j(\xi, v, \delta)$ is the greater of the utility attainable at the end of period T_j and the present valued searcher's expectation of the net utility attainable from continuing search past the end of period T_j according to the search rule $\rho = (\xi, v, \delta)$.

Section 6-2 develops the optimal stopping rule component ξ^* of ρ^* , Section 6-3 develops the optimal observation number rule component v^* and Section 6-4 develops the optimal terminal decision procedure δ^* . The results of Sections 6-2, 6-3 and 6-4 are combined in Section 6-5 to prove the optimality of ρ^* . The conditions under which Stigleresque and sequential search behaviours coincide with the more generally optimal generalised search behaviour are examined more formally in Section 6-6.

SECTION 6-2: THE OPTIMAL GENERALISED SEARCH STOPPING RULE

In Sections 1-4 and 6-1, the optimal generalised search rule is described as an application of Bellman's principle of optimality. The flavour of Bellman's principle is seen in the construction of the optimal sequential stopping rule constructed in Section 2-3. The stopping rule component ξ^* of the optimal generalised search rule ρ^* is shown in this section to also be a backward induction stopping rule. ξ^* is an adaptive stopping rule which continues search up to the point where the searcher expects that net utility cannot be increased by additional search.

The search models discussed in Chapter 2 excluded psychic search costs. The inclusion of psychic costs and the observation number rule in the generalised search model make necessary a second presentation

of the construction of the backward induction stopping rule. As in Chapter 2, it is initially assumed that only a finite number of search periods exist. This restriction is later shown to be redundant for all realistic search situations but it is temporarily useful in constructing ρ^* and proving its optimality.

Assumption: There is only a finite number J of search periods. 6-2-1

The presence of assumption (6-2-1) will be denoted by a superscript J .

For instance, a stopping rule ξ will be denoted by ξ^J and the present valued ex ante expected net utility from the use of a generalised search rule $\rho^J = (\xi^J, v^J, \delta)$ will be denoted by $V_0^J(\xi^J, v^J, \delta)$.

Definition:

ξ^{bJ} is the backward induction stopping rule when there are only $J < \infty$ search periods. 6-2-2

Definition:

ξ^{J*} is an optimal stopping rule if and only if

$$V_0^J(\xi^{J*}, v^J, \delta) \geq V_0^J(\xi^J, v^J, \delta) \quad 6-2-3$$

for any observation number rule v^J and any terminal decision procedure δ .

The construction of ξ^{bJ} is presented for any given v^J and δ .

Proposition 6-2-1 shows that, subject to assumption (6-2-1), ξ^{bJ} is an optimal stopping rule ie. the present valued ex ante expected net utility from search using ξ^{bJ} cannot be exceeded by using any other stopping rule.

Derivation of the Backward Induction Stopping Rule ξ^{bJ} :

The backward induction stopping rule ξ^{bJ} is

$$\xi^{bJ} = (\xi_0^{bJ}, \xi_1^{bJ}, \dots, \xi_{J-1}^{bJ}, \xi_J^{bJ}) \quad 6-2-4$$

$$\text{where } \xi_i^{bJ}(y_i) = \begin{cases} 0, & \text{if search is to continue through period } T_{i+1} \\ 1, & \text{if search is to stop at the end of period } T_i \end{cases} \quad 6-2-5$$

for all $i = 0, 1, \dots, J-1$. Since it is assumed that only J search periods exist, search cannot persist past the end of period T_J .

Hence,

$$\xi_J^{bJ}(y_J) = S_J^{bJ} \equiv 1 \quad 6-2-6$$

If the searcher finds himself at the end of period T_J , his vector of observed prices is $y_J = (p_1^{11}, \dots, p_1^{JnJ})$. For a given terminal decision procedure δ , the searcher's attainable utility at the end of period T_J is $I(\delta(y_J))$. By definition,

$$V_J^J(\xi^{bJ}, v^J, \delta) = I(\delta(y_J)) \quad 6-2-7$$

Suppose the searcher finds himself at the end of period T_{J-1} . The vector of his observed prices is $y_{J-1} = (p_1^{11}, \dots, p_1^{J-1, n_{J-1}})$. If he chooses to stop searching at this point the attainable utility is $I(\delta(y_{J-1}))$. If he chooses to continue his search through period T_J he must choose some number n_J of observations on p_1 to make in period T_J . The present value of the net utility the searcher expects to attain at the end of period T_J by taking n_J extra observations is

$$\begin{aligned} & (1 - \alpha)(E[I(\delta(y_J)) | y_{J-1}] - K(J, n_J)) \\ &= (1 - \alpha)(E[V_J^J(\xi^{bJ}, v^J, \delta) | y_{J-1}] - K(J, n_J)) \end{aligned} \quad 6-2-8$$

by (6-2-7). If the net utility exceeds $I(\delta(y_{J-1}))$ the searcher expects that the present value of the utility attainable by searching on through period T_J exceeds the utility presently attainable. ξ_{J-1}^{bJ}

is defined so that the searcher continues his search through period T_J if and only if this action provides a higher present valued expected net utility.

$$\xi_{J-1}^{bJ}(y_{J-1}) = S_{J-1}^{bJ} = \begin{cases} 0, & \text{if } I(\delta(y_{J-1})) < (1-\alpha)(E[V_J^J(\xi^{bJ}, v^J, \delta) | y_{J-1}] \\ & - K(J, n_J)) \\ 1, & \text{if } I(\delta(y_{J-1})) \geq (1-\alpha)(E[V_J^J(\xi^{bJ}, v^J, \delta) | y_{J-1}] \\ & - K(J, n_J)) \end{cases} \quad 6-2-9$$

By definition,

$$V_{J-1}^J(\xi^{bJ}, v^J, \delta) = \max\{I(\delta(y_{J-1})), (1-\alpha)(E[V_J^J(\xi^{bJ}, v^J, \delta) | y_{J-1}] - K(J, n_J))\} \quad 6-2-10$$

Hence, (6-2-9) is equivalent to

$$\xi_{J-1}^{bJ}(y_{J-1}) = S_{J-1}^{bJ} = \begin{cases} 0, & \text{if } I(\delta(y_{J-1})) \neq V_{J-1}^J(\xi^{bJ}, v^J, \delta) \\ 1, & \text{if } I(\delta(y_{J-1})) = V_{J-1}^J(\xi^{bJ}, v^J, \delta) \end{cases} \quad 6-2-11$$

Now suppose the searcher finds himself at the end of period T_{J-2} . The utility attainable now is $I(\delta(y_{J-2}))$. The present value of the expected net utility of making n_{J-1} observations in period T_{J-1} and n_J observations in period T_J is

$$(1-\alpha)^2 E[I(\delta(y_J)) | y_{J-2}] - (1-\alpha)K(J-1, n_{J-1}) - (1-\alpha)^2 K(J, n_J) \quad 6-2-12$$

Note that, if $n_J = 0$, then the searcher anticipates stopping his search at the end of period T_{J-1} . The searcher's expectation at the end of period T_{J-2} of the present value of the net utility attainable from search is

$$\begin{aligned}
 V_{J-2}^J(\xi^{bJ}, v^J, \delta) = & \max\{I(\delta(y_{J-2})), (1-\alpha)(E[I(\delta(y_{J-1}))|y_{J-2}] - K(J-1, n_{J-1})), \\
 & (1-\alpha)^2(E[I(\delta(y_J))|y_{J-2}] - (1-\alpha)K(J-1, n_{J-1}) \\
 & - (1-\alpha)^2K(J, n_J))\} \quad 6-2-13
 \end{aligned}$$

$$\begin{aligned}
 = & \max\{I(\delta(y_{J-2})), (1-\alpha)(\max\{E[I(\delta(y_{J-1}))|y_{J-2}], \\
 & (1-\alpha)(E[I(\delta(y_J))|y_{J-2}] - K(J, n_J))\} - K(J-1, n_{J-1}))\} \quad 6-2-14
 \end{aligned}$$

$$\begin{aligned}
 = & \max\{I(\delta(y_{J-2})), (1-\alpha)(E[V_{J-1}^J(\xi^{bJ}, v^J, \delta)|y_{J-2}] \\
 & - K(J-1, n_{J-1}))\} \quad 6-2-15
 \end{aligned}$$

ξ_{J-2}^{bJ} is defined so that the searcher continues his search through period T_{J-1} if and only if this action provides a higher present valued expected net utility.

$$\xi_{J-2}^{bJ}(y_{J-2}) = S_{J-2}^{bJ} = \begin{cases} 0, & \text{if } I(\delta(y_{J-2})) \neq V_{J-2}^J(\xi^{bJ}, v^J, \delta) \\ 1, & \text{if } I(\delta(y_{J-2})) = V_{J-2}^J(\xi^{bJ}, v^J, \delta) \end{cases} \quad 6-2-16$$

Continuing the backward induction in this manner for $j = J-3, \dots, 2, 1, 0$ shows that the present value, at the end of period T_j , of the expected net utility from search is

$$V_j^J(\xi^{bJ}, v^J, \delta) = \max\{I(\delta(y_j)), (1-\alpha)(E[V_{j+1}^J(\xi^{bJ}, v^J, \delta)|y_j] - K(j+1, n_{j+1}))\} \quad 6-2-17$$

and that the j th element of the backward induction stopping rule is

$$\xi_j^{bJ}(y_j) = S_j^{bJ} = \begin{cases} 0, & \text{if } I(\delta(y_j)) \neq V_j^J(\xi^{bJ}, v^J, \delta) \\ 1, & \text{if } I(\delta(y_j)) = V_j^J(\xi^{bJ}, v^J, \delta) \end{cases} \quad 6-2-18$$

for $j = 0, 1, \dots, J-1$. Recall that the J th element of ξ^{bJ} is defined to be identically unity by (6-2-6).

This completes the construction of the backward induction stopping rule for the generalised search model restricted by assumption (6-2-1). The proof of the optimality of the backward induction stopping rule ξ^{bJ} for the generalised search model is given by Proposition 6-2-1 and is similar to the proof of Proposition 2-5-3 which established the optimality of a backward induction stopping rule for the sequential search model. Like Proposition 2-5-3, the proof of Proposition 6-2-1 relies upon successive applications of Lemma 2-5-2.

Proposition 6-2-1:

If there is only a finite number J of search periods, then the backward induction stopping rule ξ^{bJ} is an optimal stopping rule.

Proof:

ξ^{bJ} is an optimal stopping rule if and only if

$$V_0^J(\xi^{bJ}, v^J, \delta) \geq V_0^J(\xi^J, v^J, \delta) \quad 6-2-19$$

for any stopping rule ξ^J , any observation number rule v^J and any terminal decision procedure δ . From (6-2-17) and (6-2-18),

$$V_0^J(\xi^J, v^J, \delta) = \max\{I(\delta(y_0)), (1-\alpha)(E[V_1^J(\xi^J, v^J, \delta)] - K(1, n_1))\} \quad 6-2-20$$

$$= S_0^J I(\delta(y_0)) + (1-S_0^J)(1-\alpha)(E[V_1^J(\xi^J, v^J, \delta)] - K(1, n_1)) \quad 6-2-21$$

Again from (6-2-17) and (6-2-18),

$$V_1^J(\xi^J, v^J, \delta) = \max\{I(\delta(y_1)), (1-\alpha)(E[V_2^J(\xi^J, v^J, \delta) | y_1] - K(2, n_2))\} \quad 6-2-22$$

$$= S_1^J I(\delta(y_1)) + (1-S_1^J)(1-\alpha)(E[V_2^J(\xi^J, v^J, \delta) | y_1] - K(2, n_2)) \quad 6-2-23$$

Substituting (6-2-23) into (6-2-21) gives

$$V_0^J(\xi^J, v^J, \delta) = S_0^J I(\delta(y_0)) + (1-S_0^J)(1-\alpha) (E[S_1^J I(\delta(y_1)) + \\ (1-S_1^J)(1-\alpha) (E[V_2^J(\xi^J, v^J, \delta) | y_1] - K(2, n_2))] - K(1, n_1)) \quad 6-2-24$$

Expanding (6-2-24) in this fashion up to the J th term gives the present value of the searcher's ex ante expected net utility from search according to a generalised search rule $\rho^J = (\xi^J, v^J, \delta)$ as

$$V_0^J(\xi^J, v^J, \delta) = S_0^J I(\delta(y_0)) + (1-S_0^J)(1-\alpha) (E[S_1^J I(\delta(y_1)) + \\ (1-S_1^J)(1-\alpha) (E[S_2^J I(\delta(y_2)) + \dots + \\ (1-S_{J-2}^J)(1-\alpha) (E[S_{J-1}^J I(\delta(y_{J-1})) + \\ (1-S_{J-1}^J)(1-\alpha) (E[S_J^J I(\delta(y_J)) | y_{J-1}] - K(J, n_J)) | y_{J-2}] \\ - K(J-1, n_{J-1})) \dots | y_1] - K(2, n_2))] - K(1, n_1)) \quad 6-2-25$$

Recall from (6-1-5) that the j th stopping rule component ξ_j^J is a function of y_j , $j = 1, \dots, J$, and recall from (6-2-6) that

$\xi_J^J(y_J) = S_J^J \equiv 1$. (6-2-25) can therefore be rewritten as

$$V_0^J(\xi^J, v^J, \delta) = S_0^J I(\delta(y_0)) + (1-S_0^J)(1-\alpha) \left(\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (S_1^J I(\delta(y_1)) + \right. \\ (1-S_1^J)(1-\alpha) \left(\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (S_2^J I(\delta(y_2)) + \dots + (1-S_{J-2}^J)(1-\alpha) \right. \\ \left. \left(\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (S_{J-1}^J I(\delta(y_{J-1})) + (1-S_{J-1}^J)(1-\alpha) \left(\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} I(\delta(y_J)) f_g(y_J | y_{J-1}) dy_J \right. \right. \right. \\ \left. \left. \left. - K(J, n_J) \right) f_g(y_{J-1} | y_{J-2}) dy_{J-1} - K(J-1, n_{J-1}) \right) \dots \right) f_g(y_2 | y_1) dy_2 \\ \left. \left. - K(2, n_2) \right) f_g(y_1) dy_1 - K(1, n_1) \right) \quad 6-2-26$$

$$\begin{aligned}
&= S_0^J I(\delta(y_0)) + (1-S_0^J) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_1^J \{ (1-\alpha) I(\delta(y_1)) - (1-\alpha) K(1, n_1) \} + \\
&\quad (1-S_1^J) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_2^J \{ (1-\alpha)^2 I(\delta(y_2)) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v) \} + \\
&\quad \dots (1-S_{J-2}^J) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_{J-1}^J \{ (1-\alpha)^{J-1} I(\delta(y_{J-1})) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v) \} + \\
&\quad (1-S_{J-1}^J) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \{ (1-\alpha)^J I(\delta(y_J)) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v) \} f_g(y_J | y_{J-1}) dy_J] \\
&\quad f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots] f_g(y_2 | y_1) dy_2] f_g(y_1) dy_1
\end{aligned}$$

6-2-27

The j th individual integral, $j = 0, \dots, J-1$, of the nest of integrals (6-2-27) is

$$\begin{aligned}
&\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_j^J \{ (1-\alpha)^j I(\delta(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, n_v) \} + \\
&\quad (1-S_j^J) (1-\alpha) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [\dots] f_g(y_{j+1} | y_j) dy_{j+1}] f_g(y_j | y_{j-1}) dy_j
\end{aligned}$$

6-2-28

which is an integral of the type considered in Lemma 2-5-2. Applying

Lemma 2-5-2 shows the function $\xi_j^J(y_j)$ which maximises (6-2-28) is

$\xi_j^{J*}(y_j)$ where

$$\xi_j^{J*}(y_j) = S_j^{J*} = \begin{cases} 0, & (1-\alpha)^j I(\delta(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, n_v) \neq \\ & \max\{(1-\alpha)^j I(\delta(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, n_v), \\ & (1-\alpha)^{j+1} (E[V_{j+1}^J(\xi^{J*}, v^J, \delta) | y_j] - K(j+1, n_{j+1})) - \\ & \sum_{v=1}^j (1-\alpha)^v K(v, n_v) \\ & 1, \text{ otherwise} \end{cases} \quad 6-2-29$$

$$\text{ie. } \xi_j^{J*}(y_j) = S_j^{J*} = \begin{cases} 0, & I(\delta(y_j)) \neq \max\{I(\delta(y_j)), (1-\alpha)(E[V_{j+1}^J(\xi^{J*}, v^J, \delta) | y_j] - K(j+1, n_{j+1}))\} \\ & \\ 1, & I(\delta(y_j)) = \max\{I(\delta(y_j)), (1-\alpha)(E[V_{j+1}^J(\xi^{J*}, v^J, \delta) | y_j] - K(j+1, n_{j+1}))\} \end{cases} \quad 6-2-30$$

$$\text{ie. } \xi_j^{J*}(y_j) = S_j^{J*} = \begin{cases} 0, & I(\delta(y_j)) \neq V_j^J(\xi^{J*}, v^J, \delta) \\ & \\ 1, & I(\delta(y_j)) = V_j^J(\xi^{J*}, v^J, \delta) \end{cases} \quad 6-2-31$$

Comparison of (6-2-31) and (6-2-18) shows $\xi_j^{J*} \equiv \xi_j^{bJ}$ and this is true for $j = 0, \dots, J$. Hence $\xi^{J*} \equiv \xi^{bJ}$ and therefore, from (2-5-15), (6-2-2) and (6-2-3)

$$V_0^J(\xi^J, v^J, \delta) \leq \max[I(\delta(y_0)), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)I(\delta(y_1)) - (1-\alpha)K(1, n_1), \\ \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)^2 I(\delta(y_2)) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v), \dots, \\ 6-2-32$$

$$\begin{aligned}
& \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)^{J-1} I(\delta(y_{J-1})) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v)], \\
& \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [(1-\alpha)^J I(\delta(y_J)) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v)] f_g(y_J | y_{J-1}) dy_J] \\
& f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots] f_g(y_2 | y_1) dy_2] f_g(y_1) dy_1] \\
& = V_0^J(\xi^{bJ}, v^J, \delta)
\end{aligned}$$

6-2-33

for any stopping rule ξ^J , any observation number rule v^J and any terminal decision procedure δ .

Q.E.D.

This completes the first steps in the specification of the optimal generalised search rule and in the proof of its optimality. Since ξ^{bJ} has been proved to be an optimal stopping rule, it is denoted from here on by ξ^{J*} .

$$\xi^{bJ} \equiv \xi^{J*}$$

6-2-34

The construction and proof of the optimality of ξ^{J*} have been presented for any observation number rule v^J and any terminal decision procedure δ . The utility attainable from search clearly depends upon the choice of v^J and δ . The next section constructs the optimal observation number rule v^{J*} and proves its optimality.

One of the characteristics of sequential search which distinguishes it from Stigleresque search is that a sequential search rule has a stopping rule component not subject to the restriction that $\xi_1(y_1) = S_1 \equiv 1$. The reader will have noticed several formal

similarities between the development of the optimal generalised search stopping rule and the optimal sequential search stopping rule developed in Chapter 2. The inclusion of an adaptive stopping rule in the generalised search rule is one of the two factors which causes generalised search to contain sequential search as a special case while also containing Stigleresque search as a special case. Conversely, one of the characteristics of Stigleresque search which distinguishes it from sequential search is that a Stigleresque search rule has an observation number rule not subject to the restriction that $n_1 = 1$ always. The inclusion of an adaptive observation number rule in the generalised search rule is the second factor which causes generalised search to contain both Stigleresque and sequential search as special cases. In the next section the reader will note several formal similarities between the development of the optimal generalised search observation number rule and the discussion in Chapter 5 of the determination of the optimal number of observations taken by a Stigleresque searcher.

SECTION 6-3: THE OPTIMAL GENERALISED SEARCH OBSERVATION NUMBER RULE

The optimal stopping rule ξ^{J*} has been shown to be a backward induction rule, subject to assumption (6-2-1). Subject to the same assumption, the optimal observation number rule v^{J*} is shown in this section to also be a backward induction rule.

The discussion of this section considers observation number rules only in the context of their being used with the optimal stopping rule ξ^{J*} . The objective of the generalised search problem is to maximise, with respect to ξ , v and δ , the present valued ex ante

expected net utility attainable from search. Proposition 6-2-1 proved that $V_0^J(\xi^{J*}, v^J, \delta) \geq V_0^J(\xi^J, v^J, \delta)$ for any observation number rule v^J and any terminal decision procedure δ . Therefore, in constructing the optimal observation number rule v^{J*} , it is superfluous to consider its use in conjunction with any stopping rule other than ξ^{J*} .

Definition:

v^{J*} is an optimal observation number rule if and only if

$$V_0^J(\xi^{J*}, v^{J*}, \delta) \geq V_0^J(\xi^{J*}, v^J, \delta) \quad 6-3-1$$

for any observation number rule v^J and any terminal decision procedure δ .

It is convenient to introduce the following notation.

Definition:

For any observation number rule $v^J = (v_0^J, v_1^J, \dots, v_{J-1}^J)$ the

vector \underline{v}_j^J is the vector of the last $(J-j)$ components of v^J

$$\text{ie. } \underline{v}_j^J = (v_j^J, v_{j+1}^J, \dots, v_{J-1}^J) \quad 6-3-2$$

Definition:

v^{bJ} is the backward induction observation number rule when

there are only $J < \infty$ search periods. 6-3-3

Derivation of the Backward Induction Observation Number Rule v^{bJ} :

The backward induction observation number rule v^{bJ} is

$$v^{bJ} = \underline{v}_0^{bJ} = (v_0^{bJ}, v_1^{bJ}, \dots, v_{J-1}^{bJ}) \quad 6-3-4$$

such that

$$v_j^{bJ}(y_j) = \underline{n}_{j+1}^{bJ} = (n_{j+1}^{bJ}, \dots, n_J^{bJ}), \quad j = 0, \dots, J-1 \quad 6-3-5$$

where n_i^{bJ} is the number of observations on p_i that the searcher intends, at the end of period T_j , to take in period T_i , $i = j+1, \dots, J$.

Suppose the searcher finds himself at the end of period T_{J-1} .

His vector of observed prices is $y_{J-1} = (p_1^{11}, \dots, p_1^{J-1, n_{J-1}})$. If

he stops searching at this point, the attainable utility is $I(\delta(y_{J-1}))$. If he continues on and takes n_J observations in period T_J , the present value of the net utility he expects to attain is

$$(1-\alpha)(E[I(\delta(y_J))|y_{J-1}, n_J] - K(J, n_J)) \quad 6-3-6$$

Let n'_J be such that

$$E[I(\delta(y_J))|y_{J-1}, n'_J] - K(J, n'_J) \geq E[I(\delta(y_J))|y_{J-1}, n_J] - K(J, n_J) \quad 6-3-7$$

for all $n_J \geq 0$. If

$$I(\delta(y_{J-1})) < (1-\alpha)(E[I(\delta(y_J))|y_{J-1}, n'_J] - K(J, n'_J)) \quad 6-3-8$$

then the action of highest present valued expected net utility is to continue search and take n'_J observations on p_1 in period T_J . If

$$I(\delta(y_{J-1})) \geq (1-\alpha)(E[I(\delta(y_J))|y_{J-1}, n'_J] - K(J, n'_J)) \quad 6-3-9$$

then the action of highest present valued expected net utility is to stop searching now, at the end of period T_{J-1} . v_{J-1}^{bJ} is defined as

$$v_{J-1}^{bJ}(y_{J-1}) = \frac{n_J^{bJ}}{n_J} = \begin{cases} n'_J, & \text{if } I(\delta(y_{J-1})) < (1-\alpha)(E[I(\delta(y_J))|y_{J-1}, n'_J] - K(J, n'_J)) \\ 0, & \text{if } I(\delta(y_{J-1})) \geq (1-\alpha)(E[I(\delta(y_J))|y_{J-1}, n'_J] - K(J, n'_J)) \end{cases} \quad 6-3-10$$

Comparing (6-2-8), (6-2-9) and (6-2-10) to (6-3-10) shows (6-3-10) may be written in the form of (6-2-11) as

$$v_{J-1}^{bJ}(y_{J-1}) = \frac{n_J^{bJ}}{n_J} = \begin{cases} n'_J, & \text{if } I(\delta(y_{J-1})) \neq V_{J-1}^J(\xi^{J*}, v_{J-1}^{bJ}, \delta) \\ 0, & \text{if } I(\delta(y_{J-1})) = V_{J-1}^J(\xi^{J*}, v_{J-1}^{bJ}, \delta) \end{cases} \quad 6-3-11$$

Comparison of (6-3-11) and (6-2-11) shows that

$$(i) \quad n_J^{bJ} = 0 \text{ if and only if } S_{J-1}^{J*} = 1 \quad 6-3-12$$

$$(ii) \quad n_J^{bJ} > 0 \text{ if and only if } S_{J-1}^{J*} = 0 \quad 6-3-13$$

(6-3-12) and (6-3-13) formally express the idea, advanced intuitively earlier, that if a search has persisted to the end of period T_{J-1} , then it is continued past the end of period T_{J-1} if and only if at least one observation on p_1 is to be taken in period T_J .

Now suppose the searcher finds himself at the end of period T_{J-2} . His vector of observed prices is $y_{J-2} = (p_1^{11}, \dots, p_1^{J-2}, n_{J-2})$. $I(\delta(y_{J-2}))$ is the attainable utility if the searcher stops his search at the end of period T_{J-2} . The searcher has the option of continuing his search past the end of period T_{J-2} . Suppose the searcher considers taking n_{J-1} observations on p_1 in period T_{J-1} and n_J observations on p_1 in period T_J . Note that $n_J = 0$ implies, by (6-3-12), that the searcher intends to halt his search at the end of period T_{J-1} . Note also that it is possible for $n_{J-1} = 0$ and $n_J > 0$. It will be shown that $n_{J-1} = n_J = 0$ implies $S_{J-2}^{J*} = 1$ (cf. the above comments on (6-3-12) and (6-3-13)). The present value of the expected net utility attained by taking n_{J-1} and n_J observations in periods T_{J-1} and T_J is

$$(1-\alpha)(E[\max\{I(\delta(y_{J-1})), (1-\alpha)(E[I(\delta(y_J)) | y_{J-1}, n_{J-1}, n_J] - K(J, n_J))\} | y_{J-2}, n_{J-1}] - K(J-1, n_{J-1}))$$

6-3-14

Let $n_{J-1}^{tJ} = (n_{J-1}^t, n_J^t)$ be such that

$$(1-\alpha) (E[\max\{I(\delta(y_{J-1})), (1-\alpha) (E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!]$$

$$K(J, n_J^!)) | y_{J-2}, n_{J-1}^!]$$

$$\geq (1-\alpha) (E[\max\{I(\delta(y_{J-1})), (1-\alpha) (E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!]$$

$$K(J, n_J^!)) | y_{J-2}, n_{J-1}^!]$$

6-3-15

for all $\underline{n}_{J-1}^J = (n_{J-1}^!, n_J^!)$. If

$$I(\delta(y_{J-2})) < (1-\alpha) (E[\max\{I(\delta(y_{J-1})), (1-\alpha) (E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!]$$

$$K(J, n_J^!)) | y_{J-2}, n_{J-1}^!]$$

6-3-16

then the action of highest present valued expected net utility is to continue search and to take $n_{J-1}^!$ observations on p_1 in period T_{J-1} .

If

$$I(\delta(y_{J-2})) \geq (1-\alpha) (E[\max\{I(\delta(y_{J-1})), (1-\alpha) (E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!]$$

$$K(J, n_J^!)) | y_{J-2}, n_{J-1}^!]$$

6-3-17

then the action of highest present valued expected net utility is to stop search now, at the end of period T_{J-2} . Consequently v_{J-2}^{bJ} is defined as

$$v_{J-2}^{bJ}(y_{J-2}) = \underline{n}_{J-1}^{bJ} = \begin{cases} (n_{J-1}^!, n_J^!), & \text{if } I(\delta(y_{J-2})) < (1-\alpha) (E[\max\{I(\delta(y_{J-1})), \\ & (1-\alpha) (E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!]$$

$$K(J, n_J^!)) | y_{J-2}, n_{J-1}^!]$$

6-3-1

$$(0, 0), & \text{if } I(\delta(y_{J-2})) \geq (1-\alpha) (E[\max\{I(\delta(y_{J-1})),$$

$$(1-\alpha) (E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!]$$

$$K(J, n_J^!)) | y_{J-2}, n_{J-1}^!]$$

By definition,

$$\begin{aligned} & \max\{I(\delta(y_{J-2})), (1-\alpha)(E[\max\{I(\delta(y_{J-1})), (1-\alpha)(E[I(\delta(y_J)) | y_{J-1}, n_{J-1}^!, n_J^!] - \\ & \quad K(J, n_J^!)) | y_{J-2}, n_{J-1}^!] - K(J-1, n_{J-1}^!))\} \\ & = \max\{I(\delta(y_{J-2})), (1-\alpha)(E[V_{J-1}^J(\xi^{J*}, v^{bJ}, \delta) | y_{J-1}, n_{J-1}^!, n_J^!] - \\ & \quad K(J-1, n_{J-1}^!))\} \end{aligned} \quad 6-3-19$$

$$= V_{J-2}^J(\xi^{J*}, v^{bJ}, \delta) \quad 6-3-20$$

By substituting (6-3-20) into (6-3-18), $v_{J-2}^{bJ}(y_{J-2})$ can be written as

$$v_{J-2}^{bJ}(y_{J-2}) = \underline{n}_{J-1}^{bJ} = \begin{cases} (n_{J-1}^!, n_J^!), & \text{if } I(\delta(y_{J-2})) \neq V_{J-2}^J(\xi^{J*}, v^{bJ}, \delta) \\ (0, 0), & \text{if } I(\delta(y_{J-2})) = V_{J-2}^J(\xi^{J*}, v^{bJ}, \delta) \end{cases} \quad 6-3-21$$

Continuing the backward induction in this manner for $j = J-3, \dots, 2, 1, 0$ shows that the present value, at the end of period T_j , of the expected net utility from search conducted according to the generalised search rule $\rho^J = (\xi^{J*}, v^{bJ}, \delta)$ is, for all $j = 0, 1, \dots, J-1$,

$$\begin{aligned} V_j^J(\xi^{J*}, v^{bJ}, \delta) &= \max\{I(\delta(y_j)), (1-\alpha)(E[V_{j+1}^J(\xi^{J*}, v^{bJ}, \delta) | y_j, \underline{n}_{j+1}^{bJ}] - \\ & \quad K(j+1, \underline{n}_{j+1}^{bJ}))\} \end{aligned} \quad 6-3-22$$

where $\underline{n}_{j+1}^{bJ} = \underline{n}_{j+1}^{!J} = (n_{j+1}^!, \dots, n_J^!)$ such that

$$\begin{aligned} & E[V_{j+1}^J(\xi^{J*}, v^{bJ}, \delta) | y_j, \underline{n}_{j+1}^{!J}] - K(j+1, \underline{n}_{j+1}^{!J}) \\ & \geq E[V_{j+1}^J(\xi^{J*}, v^J, \delta) | y_j, \underline{n}_{j+1}^J] - K(j+1, \underline{n}_{j+1}^J) \end{aligned} \quad 6-3-23$$

for all $\underline{n}_{j+1}^J = (n_{j+1}, \dots, n_J)$, and shows that

$$v_j^{bJ}(y_j) = n_{j+1}^{bJ} = \begin{cases} (n_{j+1}^1, \dots, n_j^1), & \text{if } I(\delta(y_j)) \neq V_j^J(\xi^{J*}, v^{bJ}, \delta) \\ (0, \dots, 0), & \text{if } I(\delta(y_j)) = V_j^J(\xi^{J*}, v^{bJ}, \delta) \end{cases} \quad 6-3-24$$

Comparison of (6-3-24) and (6-2-18) shows that

$$(i) \quad \sum_{i=j+1}^J n_i^{bJ} = 0 \text{ if and only if } \xi_j^{J*}(y_j) = S_j^{J*} = 1 \quad 6-3-25$$

$$(ii) \quad \sum_{i=j+1}^J n_i^{bJ} > 0 \text{ if and only if } \xi_j^{J*}(y_j) = S_j^{J*} = 0 \quad 6-3-26$$

This completes the construction of the backward induction observation number rule v^{bJ} for the generalised search model restricted by assumption (6-2-1). Some similarities and differences between ξ^{J*} and v^{bJ} are worthwhile considering. The method of construction of each is similar. At the end of period T_j , where $j = 0, \dots, J-1$, the searcher assesses the value S_j^{J*} of $\xi_j^{J*}(y_j)$ and the values of the observation number rule elements $(n_{j+1}^{bJ}, \dots, n_J^{bJ}) = n_{j+1}^{bJ} = v_j^{bJ}(y_j)$. If $S_j^{J*} = 1$ then, by (6-3-25), $n_{j+1}^{bJ} = (0, \dots, 0)$ since search is stopped now, at the end of period T_j . If $S_j^{J*} = 0$ then, by (6-3-26), $n_{j+1}^{bJ} \neq (0, \dots, 0)$ and $n_{j+1}^{bJ} \geq 0$ observations on p_1 are taken throughout period T_{j+1} . At the end of period T_{j+1} , the searcher assesses the value S_{j+1}^{J*} of $\xi_{j+1}^{J*}(y_{j+1})$ and the new values of the observation number rule elements $(n_{j+2}^{bJ}, \dots, n_J^{bJ}) = n_{j+2}^{bJ} = v_{j+1}^{bJ}(y_{j+1})$. This process is repeated until either the searcher chooses to halt his search or the end of the last period T_J is reached.

The proof that the backward induction observation number rule is an optimal observation number rule is given in Proposition 6-3-1

and is similar to the proof given in Proposition 6-2-1 for the optimality of the backward induction stopping rule.

Proposition 6-3-1:

If there is only a finite number J of search periods, then the backward induction observation number rule v^{bJ} is an optimal observation number rule,

Proof:

v^{bJ} is an optimal observation number rule if and only if

$$V_0^J(\xi^{J*}, v^{bJ}, \delta) \geq V_0^J(\xi^{J*}, v^J, \delta) \quad 6-3-27$$

for any observation number rule v^J and any terminal decision procedure δ ,

From (6-2-33),

$$V_0^J(\xi^{J*}, v^J, \delta) = \max[I(\delta(y_0)), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)I(\delta(y_1)) - (1-\alpha)K(1, n_1),$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)^2 I(\delta(y_2)) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v), \dots,$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)^{J-1} I(\delta(y_{J-1})) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v),$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [(1-\alpha)^J I(\delta(y_J)) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v)]$$

$$f_g(y_J | y_{J-1}) dy_J] f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots]$$

$$f_g(y_2 | y_1) dy_2] f_g(y_1) dy_1]$$

$$\leq \max[I(\delta(y_0)), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max\{(1-\alpha)I(\delta(y_1)) - (1-\alpha)K(1, n_1),$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max\{(1-\alpha)^2 I(\delta(y_2)) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v), \dots, \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[$$

$$\max\{(1-\alpha)^{J-1} I(\delta(y_{J-1})) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max\{(1-\alpha)^J$$

$$I(\delta(y_J)) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v)\} f_g(y_J | y_{J-1}) dy_J \} f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots \}$$

$$f_g(y_2 | y_1) dy_2 \} f_g(y_1) dy_1]$$

6-3-28

$$= \max[I(\delta(y_0)), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)I(\delta(y_1)) - (1-\alpha)K(1, n_1^{bJ}),$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[(1-\alpha)^2 I(\delta(y_2)) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v^{bJ}), \dots, \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[$$

6-3-29

$$(1-\alpha)^{J-1} I(\delta(y_{J-1})) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v^{bJ}), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [(1-\alpha)^J I(\delta(y_J)) -$$

$$\sum_{v=1}^J (1-\alpha)^v K(v, n_v^{bJ})] f_g(y_J | y_{J-1}) dy_J \} f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots]$$

$$f_g(y_2 | y_1) dy_2 \} f_g(y_1) dy_1]$$

$$= V_0^J(\xi^{J*}, v^{bJ}, \delta)$$

6-3-30

by (6-3-22) and (6-3-23).

Q.E.D.

This completes the second step in the specification of the optimal generalised search rule ρ^* and in the proof of its optimality. Since v^{bJ} has been proved to be an optimal observation number rule it is denoted from here on by v^{J*} ie.

$$v^{bJ} \equiv v^{J*}$$

6-3-31

The third and last step in the specification of ρ^* is the specification of the optimal terminal decision procedure δ^* . This is the subject of Section 6-4.

SECTION 6-4: THE OPTIMAL GENERALISED SEARCH TERMINAL DECISION PROCEDURE

The present valued ex ante expected net utility from search conducted according to the optimal stopping rule ξ^{J*} and the optimal observation number rule v^{J*} is dependent upon the terminal decision procedure δ used. The objective in constructing an optimal generalised search rule is to describe a search behaviour which the searcher expects ex ante to yield him the greatest present valued net utility from search. Consequently, an optimal terminal decision procedure δ^* need be discussed only in the context of a generalised search rule which uses an optimal stopping rule ξ^{J*} and an optimal observation number rule v^{J*} since Propositions 6-2-1 and 6-3-1 established that

$$V_0^J(\xi^{J*}, v^{J*}, \delta) \geq V_0^J(\xi^J, v^J, \delta) \quad 6-4-1$$

for any generalised search rule $\rho^J = (\xi^J, v^J, \delta)$. An optimal terminal decision procedure δ^* is therefore defined as having the property

of maximising the present value of the ex ante expected net utility from search when used in conjunction with ξ^{J^*} and v^{J^*} .

Definition:

δ^* is an optimal terminal decision procedure if and only if

$$V_0^J(\xi^{J^*}, v^{J^*}, \delta^*) \geq V_0^J(\xi^{J^*}, v^{J^*}, \delta) \quad 6-4-2$$

for any terminal decision procedure δ .

In Chapter 2, where the search model was restricted to sequential search and zero-one demand for commodity 1, δ^* was shown to choose between two types of terminal action. The searcher either purchased zero units of commodity 1 or else he purchased one unit of commodity 1 from the seller whose combined quoted selling price and transaction cost was the smallest observed. In simpler previous consumer search models (eg. [43], [50]), the first terminal action is assumed to never be optimal so that demand for commodity 1 is fixed at one unit. In this case the optimal terminal decision procedure was shown to be equivalent to a procedure with an objective of minimising the total expenditure on commodity 1. When demand is continuously variable with respect to prices and wealth, such an expenditure minimising terminal decision procedure is usually not optimal. Whether or not expenditure on commodity 1 rises or falls with changes in wealth and the price of commodity 1 depends on the consumer's income and price elasticities of demand for commodity 1.

Proposition 6-4-1 establishes what one would expect from applying Bellman's principle of optimality to the problem of choosing a terminal decision procedure. δ^* is the terminal decision procedure which, if search halts at the end of period T_j , selects the terminal action $a^* \in A_j$ which maximises the utility attainable at the end of period T_j . The form of δ^* depends upon the nature of the costs

faced by the searcher and, in general, there is no simple way of expressing δ^* as a function of costs and observed prices. However, Proposition 6-4-2 shows that if transaction costs are the same for all sellers, then δ^* has a simpler form - if δ^* does not select action a_0 , to purchase zero units of commodity 1, then δ^* selects the terminal action of purchasing the quantity of commodity 1 demanded from the seller who offered the lowest price.

Proposition 6-4-1:

The optimal terminal decision procedure δ^* is such that, if search halts at the end of period T_j for any $j = 0, \dots, J$, then $\delta^*(y_j) = a^*$ where $I(a^*) \geq I(a)$ for all terminal actions $a \in A_j$.

Proof:

δ^* is an optimal terminal decision procedure if and only if

$$V_0^J(\xi^{J*}, v^{J*}, \delta^*) \geq V_0^J(\xi^{J*}, v^{J*}, \delta) \quad 6-4-3$$

for any terminal decision procedure δ . δ^* is such that

$$I(\delta^*(y_j)) \geq I(a) \text{ for all } a \in A_j, \text{ for any } j = 0, \dots, J.$$

Hence $I(\delta^*(y_j)) \geq I(\delta(y_j))$ for any $j = 0, \dots, J$ and for 6-4-4

any terminal decision procedure δ . From (6-3-28), for any δ ,

$$V_0^J(\xi^{J*}, v^{J*}, \delta) = \max[I(\delta(y_0)), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{n_1^J} \{ (1-\alpha) I(\delta(y_1)) -$$

$$(1-\alpha)K(1, n_1), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{n_2^J} \{ (1-\alpha)^2 I(\delta(y_2)) -$$

$$\sum_{v=1}^2 (1-\alpha)^v K(v, n_v), \dots, \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{\underline{n}_{J-1}^J} \{ (1-\alpha)^{J-1} I(\delta(y_{J-1})) -$$

$$\sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v), \dots, \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [\max\{ (1-\alpha)^J I(\delta(y_J)) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v) \}]$$

$$f_g(y_J | y_{J-1}) dy_J \} \dots \{ f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots \} \{ f_g(y_2 | y_1) dy_2 \} \{ f_g(y_1) dy_1 \} \quad 6-4-5$$

$$\leq \max[I(a_0), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{a \in A_1} \{ (1-\alpha) \max I(a) - (1-\alpha) K(1, n_1),$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{a \in A_2} \{ (1-\alpha)^2 \max I(a) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v), \dots,$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{a \in A_{J-1}} \{ (1-\alpha)^{J-1} \max I(a) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v),$$

6-4-6

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [\max\{ (1-\alpha)^J \max_{a \in A_J} I(a) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v) \}] f_g(y_J | y_{J-1}) dy_J \}$$

$$f_g(y_{J-1} | y_{J-2}) dy_{J-1} \dots \{ f_g(y_2 | y_1) dy_2 \} \{ f_g(y_1) dy_1 \}$$

$$= \max[I(\delta^*(y_0)), \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max[\max_{\underline{n}_1^J} \{ (1-\alpha) I(\delta^*(y_1)) - (1-\alpha) K(1, n_1),$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max_{n_2^J} [\max \{ (1-\alpha)^2 I(\delta^*(y_2)) - \sum_{v=1}^2 (1-\alpha)^v K(v, n_v), \dots,$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max_{n_{J-1}^J} [\max \{ (1-\alpha)^{J-1} I(\delta^*(y_{J-1})) - \sum_{v=1}^{J-1} (1-\alpha)^v K(v, n_v), \dots, \quad 6-4-7$$

$$\int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \max_{n_J^J} [\max \{ (1-\alpha)^J I(\delta^*(y_J)) - \sum_{v=1}^J (1-\alpha)^v K(v, n_v) \}]$$

$$f_g(y_J | y_{J-1}) dy_J \} \dots \} f_g(y_2 | y_1) dy_2 \} f_g(y_1) dy_1]$$

$$= V_0^J(\xi^{J*}, v^{J*}, \delta^*) \quad 6-4-8$$

by (6-4-4) and (6-3-28).

Q.E.D.

At the end of period T_j the searcher has made a total of $\sum_{i=1}^j n_i^{J*}$ observations on p_1 and his wealth, net of financial search costs, is $\bar{M}_j = M - c(j)$. δ^* selects the utility maximising terminal action a^* from the $(\sum_{i=1}^j n_i^{J*} + 1)$ terminal actions contained in the terminal action space A_j .

Suppose the optimal terminal action is a_0 . Then the searcher allocates wealth of \bar{M}_j to purchases of commodities 2 to ℓ . The optimality of action a_0 requires that the prices observed for commodity 1 are all such that the searcher willingly demands none of commodity 1 from any of the sellers contacted during search. The price p_{1j}^* is defined as the smallest price for commodity 1

for which a consumer with wealth \bar{M}_j will demand none of commodity 1.

Definition:

$$p_{1j}^* \text{ is such that } x_1(p_1, p_2, \dots, p_\ell, \bar{M}_j) = 0 \text{ if } p_1 \geq p_{1j}^* \quad 6-4-9$$

$$> 0 \text{ if } p_1 < p_{1j}^*$$

It should be noted that (6-4-9) tacitly assumes commodity 1 is not a Giffen good. The utility attained by taking action a_0 at the end of period T_j is

$$I(a_0) = I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j) \quad 6-4-10$$

Suppose the optimal terminal action is a_{ik} . Then the searcher allocates wealth of $(\bar{M}_j - t_{ik})$ to purchases of commodities 1 to ℓ . The optimality of action a_{ik} requires that the price for commodity 1 offered by the k th seller contacted in the i th period was such that the searcher willingly demands a positive quantity of commodity 1 from him. The utility attained by taking action a_{ik} at the end of period T_j is

$$I(a_{ik}) = I(p_1^{ik}, p_2, \dots, p_\ell, \bar{M}_j - t_{ik}) \quad 6-4-11$$

The utility yielded by the optimal terminal decision procedure δ^* at the end of period T_j is, therefore,

$$I(a^*) = \max\{I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j), I(p_1^{ik}, p_2, \dots, p_\ell, \bar{M}_j - t_{ik}),$$

$$\text{for all } i, k\} \quad 6-4-12$$

δ^* thus determines if any of commodity 1 is to be purchased and, if so, also determines the quantity purchased and the seller from whom the purchase is made.

If transaction costs vary between sellers, there seems to be no simpler way of expressing δ^* than as in (6-4-12). However, if transaction costs are the same for all sellers, a much simpler way of expressing δ^* is available. This form is the result of Proposition 6-4-2.

Assumption: All sellers of commodity 1 have the same transaction costs ie,

$$t_{ik} = t \geq 0 \text{ for all } i, k, \quad 6-4-13$$

The result of Proposition 6-4-2 incorporates a variable p_{1j}^{**} which is defined as the price for commodity 1 for which the consumer is indifferent between terminal action a_0 and any other terminal action $a_{ik} \in A_j$.

Definition:

$$p_{1j}^{**} \text{ is such that } I(p_{1j}^{**}, p_2, \dots, p_\ell, \bar{M}_j - t) = I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j)$$

6-4-14

Since $(\bar{M}_j - t) \leq \bar{M}_j$, (5-2-7), (5-2-9) and (6-4-14) imply that

$$p_{1j}^{**} \leq p_{1j}^* \quad 6-4-15$$

p_{1j}^{**} is the price for commodity 1 for which a consumer faced with a transaction cost of t is indifferent between demanding zero units of commodity 1 and $x_1(p_{1j}^{**}, p_2, \dots, p_\ell, \bar{M}_j - t)$ units of commodity 1. Note that demand for commodity 1 is jump discontinuous with respect to p_1 when $p_1 = p_{1j}^{**}$, the size and the position of the discontinuity depending upon the size of the transaction cost t .

Proposition 6-4-2:

$$\delta^*(y_j) = \begin{cases} a_{ik}, & \text{if } p_1^{ik} = p_{1j}^{\min} < p_{1j}^{**} \\ a_0, & \text{if } p_{1j}^{**} < p_{1j}^{\min} \end{cases}$$

for all $j = 1, 2, \dots, J$,

Proof:

(a) Suppose the vector of observed prices, y_j , is such that

$$\delta^*(y_j) = a_{ik}. \text{ Then, from (6-4-12),}$$

$$I(p_1^{ik}, p_2, \dots, p_\ell, \bar{M}_j - t) > \max\{I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j), I(p_1^{rs}, p_2, \dots, p_\ell, \bar{M}_j - t)\}$$

6-4-16

for all $s = 1, \dots, n_r$ and all $r = 1, \dots, j$,

$$\text{Since } I(p_1^{ik}, p_2, \dots, p_\ell, \bar{M}_j - t) > I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j),$$

$$I(p_1^{ik}, p_2, \dots, p_\ell, \bar{M}_j - t) > I(p_{1j}^{**}, p_2, \dots, p_\ell, \bar{M}_j - t) \quad 6-4-17$$

by (6-4-14). $\frac{\partial I}{\partial p_1} < 0$ by (5-2-7), so (6-4-17) implies

$$p_1^{ik} < p_{1j}^{**} \quad 6-4-18$$

$$\text{Since } I(p_1^{ik}, p_2, \dots, p_\ell, \bar{M}_j - t) > I(p_1^{rs}, p_2, \dots, p_\ell, \bar{M}_j - t),$$

$$p_1^{ik} < p_1^{rs}, \text{ by (5-2-7), for all } r, s. \text{ Hence}$$

$$p_1^{ik} = \min p_{1j} \quad 6-4-19$$

Combining (6-4-18) and (6-4-19) proves that

$$\delta^*(y_j) = a_{ik} \text{ if } p_1^{ik} = \min p_{1j} < p_{1j}^{**} \quad 6-4-20$$

(b) Suppose the vector of observed prices, y_j , is such that

$$\delta^*(y_j) = a_0. \text{ Then, from (6-4-12),}$$

$$I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j) > I(p_1^{rs}, p_2, \dots, p_\ell, \bar{M}_j - t) \quad 6-4-21$$

for all $s = 1, \dots, n_r$ and all $r = 1, \dots, j$. In particular,

$$I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j) > I(p_{1j}^{\min}, p_2, \dots, p_\ell, \bar{M}_j - t) \quad 6-4-22$$

Comparing (6-4-14) and (6-4-22) shows

$$I(p_{1j}^{**}, p_2, \dots, p_\ell, \bar{M}_j - t) > I(p_{1j}^{\min}, p_2, \dots, p_\ell, \bar{M}_j - t) \quad 6-4-23$$

(5-2-7) and (6-4-23) together imply

$$p_{1j}^{**} < p_{1j}^{\min} \quad 6-4-24$$

Combining (6-4-20) and (6-4-24) proves the desired result.

Q.E.D.

An examination of (6-4-14) shows that $p_{1j}^{**} \rightarrow p_{1j}^*$ as $t \rightarrow 0$.

In the absence of transaction costs, therefore, the result of

Proposition 6-4-2 becomes

$$\delta^*(y_j) = \begin{cases} a_{ik}, & \text{if } p_1^{ik} = p_{1j}^{\min} < p_{1j}^* \\ a_0, & \text{if } p_{1j}^* < p_{1j}^{\min} \end{cases} \quad 6-4-25$$

If commodity 1 is not a Giffen good, as implied by (6-4-9), then, since $\frac{\partial x_1}{\partial p_1} < 0$ for all $p_1 \in [p_1^L, p_1^U]$, the terminal decision procedure can be re-expressed in terms of the consumer's demand for commodity 1. This re-expression of δ^* is presented in Corollary 6-4-3 which shows that, if the optimal terminal action is not a_0 , then it is to patronise the seller from whom the searcher would demand the greatest quantity of commodity 1.

Corollary 6-4-3:

$$\delta^*(y_j) = \begin{cases} a_{ik}, & \text{if } \max_{r,s} x_1^*(p_1^{rs}) = x_1^*(p_1^{ik}) > 0 \\ a_0, & \text{if } x_1^*(p_1^{rs}) = 0 \end{cases}$$

for all $s = 1, \dots, n_T^{J*}$, and all $r = 1, \dots, j$.

Proof:

From Proposition 6-4-2, if $\delta^*(y_j) = a_{ik}$, then $p_1^{ik} = p_{1j}^{\min} < p_{1j}^{**}$.

$\frac{\partial x_1}{\partial p_1} < 0$ implies $x_1^*(p_1^{ik}) = x_1^*(p_{1j}^{\min}) \geq x_1^*(p_1^{rs})$ for all r, s .

By definition, $x_1^*(p_1) = 0$ for $p_1 \leq p_{1j}^{**}$
 > 0 for $p_1 > p_{1j}^{**}$

Since $p_1^{ik} < p_{1j}^{**}$, $x_1^*(p_1^{ik}) > 0$. Hence,

$\delta^*(y_j) = a_{ik}$ if $\max_{r,s} x_1^*(p_1^{rs}) = x_1^*(p_1^{ik}) > 0$.

From Proposition 6-4-2, if $\delta^*(y_j) = a_0$, then $p_{1j}^{\min} > p_{1j}^{**}$.

$\frac{\partial x_1}{\partial p_1} < 0$ implies $x_1^*(p_{1j}^{**}) = 0 = x_1^*(p_{1j}^{\min}) = x_1^*(p_1^{rs})$

for all r, s . Hence,

$\delta^*(y_j) = a_0$ if $x_1^*(p_1^{rs}) = 0$ for all r, s

Q.E.D.

Two of the effects of transaction costs on demand are worth noting. First, for any given lowest observed price $p_{1j}^{\min} < p_{1j}^*$, demand for commodity 1 is lower if $t > 0$ than if $t = 0$. Transaction costs thus lower the demand for commodity 1. Second, $x_1(p_{1j}^{**}, \bar{M}_j - t) > 0$ so no quantity of commodity 1 smaller than $x_1(p_{1j}^{**}, \bar{M}_j - t)$ will ever be purchased. Transaction costs thus place a lower bound on the number of units of commodity 1 exchanged in any one transaction.

The three components of the generalised search rule $\rho^{J*} = (\xi^{J*}, v^{J*}, \delta^*)$ have been constructed and their properties

described. The next section establishes that ρ^{J*} is optimal in that a searcher expects ex ante that its use will maximise his present valued net utility from search,

SECTION 6-5: THE OPTIMAL GENERALISED SEARCH RULE

Sections 6-2, 6-3 and 6-4 presented the components of the search rule $\rho^{J*} = (\xi^{J*}, v^{J*}, \delta^*)$ and proved the individual optimalities of these components, subject to assumption (6-2-1) that there is only a finite number J of search periods. Proposition 6-5-1 combines the results of Propositions 6-2-2, 6-3-1 and 6-4-1 to show ρ^{J*} is an optimal generalised search rule, subject to assumption (6-2-1). As in Section 2-5, where assumption (2-4-1) was shown to be redundant, it is shown in this section that assumption (6-2-1) is very weak. In any realistic search problem the conditions of search are such that (6-2-1) will be satisfied and, therefore, be redundant. Examples of search conditions sufficient to satisfy (6-2-1) are given after Proposition 6-5-1.

Proposition 6-5-1:

If search will not persist for more than $J < \infty$ search periods, the generalised search rule $\rho^{J*} = (\xi^{J*}, v^{J*}, \delta^*)$ is optimal.

Proof:

By (6-1-7), $\rho^{J*} = (\xi^{J*}, v^{J*}, \delta^*)$ is an optimal search rule if and only if

$$V_0^J(\xi^{J*}, v^{J*}, \delta^*) \geq V_0^J(\xi^J, v^J, \delta) \quad 6-5-1$$

for any generalised search rule $\rho^J = (\xi^J, v^J, \delta)$. Proposition 6-4-1 shows that

$$V_0^J(\xi^{J*}, v^{J*}, \delta^*) \geq V_0^J(\xi^{J*}, v^{J*}, \delta) \quad 6-5-2$$

for any terminal decision procedure δ . Proposition 6-3-1 shows that

$$V_0^J(\xi^{J^*}, v^{J^*}, \delta) \geq V_0^J(\xi^{J^*}, v^J, \delta) \quad 6-5-3$$

for any observation number rule v^J and any terminal decision procedure δ . Proposition 6-2-1 shows that

$$V_0^J(\xi^{J^*}, v^J, \delta) \geq V_0^J(\xi^J, v^J, \delta) \quad 6-5-4$$

for any stopping rule ξ^J , any observation number rule v^J and any terminal decision procedure δ . Combining (6-5-2), (6-5-3) and (6-5-4) proves (6-5-1).

Q.E.D.

It should be realised that, in the context of generalised search, finiteness in the quantity of search undertaken implies not only that search persists over only a finite number of periods, but also that only a finite number of observations on p_1 are taken in any particular period. The term "search length", which is used freely in the search literature, is thus ambiguous in the context of generalised search. One must distinguish carefully between the number of observations taken and the number of periods over which observations are taken.

Propositions 6-5-2 and 6-5-3 prove that conditions (6-5-5) and (6-5-6) are sufficient for search to persist over only a finite number of periods.

$$\lim_{j \rightarrow \infty} c(j) \geq M \quad 6-5-5$$

$$\sum_{v=J^{**}+1}^j (1-\alpha)^{v-J^{**}} K(v, n_v^*) > (1-\alpha)^{j-J^{**}} I(p_1^L, p_2, \dots, p_\ell, M) \text{ for all } j > J^{**}, J^{**} < \infty \quad 6-5-6$$

(6-5-5) requires that financial search costs accumulated over successive

periods eventually drive the searcher into bankruptcy. (6-5-6) requires that discounted psychic search costs accumulated past some period T_{j**} exceed the greatest present valued utility that can possibly be gained from search. Clearly one or both of these conditions will always apply in reality so that, in reality, assumption (6-2-1) is redundant.

Propositions 6-5-5 and 6-5-6 prove that conditions (6-5-7) and (6-5-8) are sufficient for only a finite number of observations on p_1 to be taken in any one period $T_j, j \geq 1$.

$$\lim_{n_j \rightarrow \infty} \sum_{i=1}^{n_j} c_{ij} \geq \bar{M}_{j-1} \text{ for all } j \geq 1 \quad 6-5-7$$

$$\lim_{n_j \rightarrow \infty} K(j, n_j) > I(p_1^L, p_2, \dots, p_L, M) \text{ for all } j \geq 1 \quad 6-5-8$$

Condition (6-5-7) is similar to condition (6-5-5) in that it requires that financial search costs accumulated within any one period eventually drive the searcher into bankruptcy. Condition (6-5-8) is similar to condition (6-5-6) in that it requires the psychic costs accumulated within any one period eventually exceed the greatest utility that can possibly be gained from search. Conditions (6-5-7) and (6-5-8) are only necessary in the limiting case of a searcher being physically able to make an infinite number of observations on p_1 within a period. Clearly this is unrealistic and an assumption of a finite intensity of search achieves the same end of guaranteeing that only a finite number of observations are taken in any one period. In reality, therefore, the number of observations

taken in any one period is restricted to being finite by whichever of these three constraints becomes binding first - (6-5-7), (6-5-8) or the physical constraint of a finite intensity of search.

Proposition 6-5-2:

If $\lim_{j \rightarrow \infty} c(j) \geq M$, then optimal search persists for only a finite number of periods.

Proof:

$\lim_{j \rightarrow \infty} c(j) \geq M$ implies that

$$\lim_{j \rightarrow \infty} \bar{M}_j = \lim_{j \rightarrow \infty} (M - c(j)) \leq 0 \quad 6-5-9$$

Let T_{J^*} be the last period for which the searcher's wealth net of financial search costs exceeds the transaction cost ie.

$$\bar{M}_{J^*} - t > 0 \quad 6-5-10$$

and

$$\bar{M}_j - t \leq 0, \text{ for all } j > J^* \quad 6-5-11$$

(6-5-10) and (6-5-11) imply

$$\bar{M}_{J^*} > \bar{M}_j, \text{ for all } j > J^* \quad 6-5-12$$

From (6-5-11), the utility attainable in period T_j , $j > J^*$, is

$$I(\delta^*(y_j)) = I(a_0) = I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j) \quad 6-5-13$$

since the searcher no longer has the net wealth to take any terminal action other than a_0 . By (6-5-12) and (5-2-9)

$$I(p_{1J^*}^*, p_2, \dots, p_\ell, \bar{M}_{J^*}) > I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j) \text{ for all } j > J^* \quad 6-5-14$$

The utility attainable at the end of period T_{J^*} is

$$I(\delta^*(y_{J^*})) = \max\{I(p_{1J^*}^*, p_2, \dots, p_\ell, \bar{M}_{J^*}), I(p_{1J^*}^{\min}, p_2, \dots, p_\ell, \bar{M}_{J^*} - t)\}$$

6-5-15

$$\geq I(p_{1J^*}^*, p_2, \dots, p_\ell, \bar{M}_{J^*})$$

$$> \max_{j > J^*} I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j)$$

$$\geq \max_{j > J^*} \{ (1-\alpha)^{j-J^*} I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j) - \sum_{v=J^*+1}^j (1-\alpha)^{v-J^*} K(v, n_v^*) \}$$

6-5-16

$$= V_{J^*+1}(\xi^*, v^*, \delta^*)$$

6-5-17

(6-5-17) guarantees that $\xi_{J^*}^*(y_{J^*}) = S_{J^*}^* = 1$ always. Hence J^* is an upper bound on the number of periods over which optimal search can persist.

Q.E.D.

Proposition 6-5-3:

$$\text{If } \sum_{v=J^{**}+1}^j (1-\alpha)^{v-J^{**}} K(v, n_v^*) > (1-\alpha)^{j-J^{**}} I(p_1^L, p_2, \dots, p_\ell, M) \text{ for}$$

all $j > J^{**}$, then optimal search persists for only a finite number of periods.

Proof:

At the end of period $T_{J^{**}}$ the searcher's present valued expected net utility from search is

$$V_{J^{**}}(\xi^*, v^*, \delta^*) = \max\{I(\delta^*(y_{J^{**}})), (1-\alpha)(E[V_{J^{**}+1}(\xi^*, v^*, \delta^*) | y_{J^{**}}] -$$

$$K(J^{**}+1, n_{J^{**}+1}^*))\}$$

6-5-18

$$I(\delta^*(y_{J^{**}})) = \max\{I(p_{1J^{**}}^*, p_2, \dots, p_\ell, \bar{M}_{J^{**}}), I(p_{1J^{**}}^{\min}, p_2, \dots, p_\ell, \bar{M}_{J^{**}-t})\}$$

6-5-19

$$\geq 0$$

6-5-20

$$(1-\alpha)(E[V_{J^{**}+1}(\xi^*, v^*, \delta^*) | y_{J^{**}}] - K(J^{**}+1, n_{J^{**}+1}^*))$$

$$= \max_{j > J^{**}} \{ (1-\alpha)^{j-J^{**}} I(\delta^*(y_j)) - \sum_{v=J^{**}+1}^j (1-\alpha)^{v-J^{**}} K(v, n_v^*) \} \quad 6-5-21$$

$$\leq \max_{j > J^{**}} \{ (1-\alpha)^{j-J^{**}} I(p_1^L, p_2, \dots, p_\ell, M) - \sum_{v=J^{**}+1}^j (1-\alpha)^{v-J^{**}} K(v, n_v^*) \} < 0$$

6-5-22

Comparing (6-5-22) and (6-5-20) shows

$$V_{J^{**}}(\xi^*, v^*, \delta^*) = I(\delta^*(y_{J^{**}})) \quad 6-5-23$$

(6-5-23) guarantees that $\xi_{J^{**}}^*(y_{J^{**}}) = S_{J^{**}}^* = 1$ always. Hence J^{**} is an upper bound on the number of periods over which search can persist.

Q.E.D.

Propositions 6-5-2 and 6-5-3 show that, in reality, search persists over only a finite number J of periods. Proposition 6-5-1 shows that, if J is finite, then the generalised search rule derived subject to assumption (6-2-1), $\rho^{J^*} = (\xi^{J^*}, v^{J^*}, \delta^*)$, is optimal. Consequently, ρ^{J^*} is an optimal generalised search rule under realistic search conditions. This is the result of Proposition 6-5-4.

Proposition 6-5-4:

If (i) $\lim_{j \rightarrow \infty} c(j) \geq M$

and/or (ii) $\sum_{v=J^{**}+1}^j (1-\alpha)^{v-J^{**}} K(v, n_v^*) > (1-\alpha)^{j-J^{**}} I(p_1^L, p_2, \dots, p_\ell, M)$

for all $j > J^{**}$, $J^{**} < \infty$,

then the generalised search rule $\rho^{J*} = (\xi^{J*}, v^{J*}, \delta^*)$ is optimal even if the number of search periods J is infinite.

Proof:

If either of (i) or (ii) apply, then Propositions 6-5-2 and 6-5-3 show $J = \min(J^*, J^{**})$ is an upper bound on the number of periods over which search will persist. Proposition 6-5-1 shows ρ^{J*} is optimal if search will persist over no more than J periods. Hence, ρ^{J*} is optimal if (i) and/or (ii) apply.

Q.E.D.

Assumption (6-2-1), that there is only a finite number J of search periods, is proved redundant in reality by Proposition 6-5-4. The superscript J is accordingly removed from the notation. The optimal search rule is denoted by $\rho^* = (\xi^*, v^*, \delta^*)$ and the accompanying present valued ex ante expected net utility from search by $V_0(\xi^*, v^*, \delta^*)$.

Propositions 6-5-5 and 6-5-6 complete the proof that, in reality, only a finite quantity of search is undertaken by a searcher using an optimal generalised search rule ρ^* . They prove that, even if the intensity of search can be infinitely great, conditions (6-5-7) and (6-5-8) guarantee only a finite number of observations on p_1 are ever taken in any one period.

Proposition 6-5-5:

If $\lim_{n_j \rightarrow \infty} \sum_{i=1}^{n_j} c_{ij} \geq \bar{M}_{j-1}$ for all $j \geq 1$, then optimal search permits

no more than a finite number of observations on p_1 to be taken in any period.

Proof:

Let $\bar{M}_j(n_j) = \bar{M}_{j-1} + \sum_{i=1}^{n_j} c_{ij}$

6-5-24

$\lim_{n_j \rightarrow \infty} \sum_{i=1}^{n_j} c_{ij} \geq \bar{M}_{j-1}$ implies that

$$\lim_{n_j \rightarrow \infty} \bar{M}_j(n_j) = \lim_{n_j \rightarrow \infty} (\bar{M}_{j-1} - \sum_{i=1}^{n_j} c_{ij}) \leq 0 \quad 6-5-25$$

Let N_j^* be the greatest number of observations the searcher can take in period T_j for which his wealth net of financial search costs exceeds the transaction cost t ie,

$$\bar{M}_j(N_j^*) - t > 0 \quad 6-5-26$$

and

$$\bar{M}_j(n_j) - t \leq 0 \text{ for all } n_j > N_j^* \quad 6-5-27$$

Hence, from (6-5-26) and (6-5-27),

$$\bar{M}_j(N_j^*) > \bar{M}_j(n_j) \text{ for all } n_j > N_j^* \quad 6-5-28$$

The utility attainable if $n_j > N_j^*$ observations on p_1 are taken is

$$I(\delta^*(y_{j-1}, p_1^{j1}, \dots, p_1^{jn_j})) = I(a_0) = I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j(n_j)) \quad 6-5-29$$

since, by (6-5-27), the searcher no longer has the net wealth to take any action other than a_0 if he takes $n_j > N_j^*$ observations on p_1 .

$$\text{However, } I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j(N_j^*)) > I(p_{1j}^*, p_2, \dots, p_\ell, \bar{M}_j(n_j)) \quad 6-5-30$$

for all $n_j > N_j^*$ by (6-5-28). Hence taking N_j^* observations on p_1 is always preferred to taking $n_j > N_j^*$ observations on p_1 . Therefore N_j^* is a finite upper bound on the number of observations taken in period T_j .

Q.E.D.

Proposition 6-5-6:

If $\lim_{n_j \rightarrow \infty} K(j, n_j) > I(p_1^L, p_2, \dots, p_\ell, M)$ for all $j \geq 1$, then optimal

search permits no more than a finite number of observations on p_1 to

be taken in any period.

Proof:

$\lim_{n_j \rightarrow \infty} K(j, n_j) > I(p_1^L, p_2, \dots, p_\ell, M)$ implies that there exists a

finite number N_j^{**} such that

$$K(j, n_j) > I(p_1^L, p_2, \dots, p_\ell, M) \text{ for all } n_j > N_j^{**} \quad 6-5-31$$

$I(p_1^L, p_2, \dots, p_\ell, M)$ is an upper bound on the utility attainable from search. Hence the searcher will prefer to take no observations on p_1 to taking $n_j > N_j^{**}$ observations on p_1 . N_j^{**} is therefore a finite upper bound on the number of observations on p_1 that will be taken in period T_j , $j \geq 1$.

Q.E.D.

Proposition 6-5-7:

If either or both of (6-5-5) and (6-5-6) are satisfied and if either or both of (6-5-7) and (6-5-8) are satisfied, then a searcher using an optimal generalised search rule ρ^* will take only a finite number of observations on p_1 throughout the whole of his search.

Proof:

If either or both of (6-5-5) and (6-5-6) are satisfied then, by Propositions 6-5-2 and 6-5-3, optimal generalised search persists over only a finite number of periods. If either or both of (6-5-7) and (6-5-8) are satisfied then, by Propositions 6-5-5 and 6-5-6, only a finite number of observations on p_1 are taken in any one period. Hence, in total, only a finite number of observations on p_1 are taken throughout the entire search.

Q.E.D.

This concludes the proofs of the optimality of ρ^* and the finiteness of the quantity of search undertaken using ρ^* .

So far the thesis has explained the construction of ρ^* and proved its optimality. The next section considers the forms which ρ^* may take under various conditions of search. The search conditions examined in particular are those under which the optimal search behaviour specified by ρ^* coincides with the behavioural extremes of Stigleresque and sequential search. This thesis has claimed that the conflict between the optimalities of the Stigleresque and sequential search rules is a consequence only of different authors analysing different search models. Both types of models have been examined and extended by this thesis. It is fitting that the last substantive section of this thesis should use the framework of generalised search to demonstrate the differences between these models, and to show how they blend into their common middle ground of generalised search as the different constraints which give rise to them are relaxed.

SECTION 6-6: THE RELATIONSHIP OF GENERALISED SEARCH TO STIGLERESQUE SEARCH AND SEQUENTIAL SEARCH

In Section 1-4 Stigleresque and sequential search rules are described as being optimal in search problems where the conditions of search are such that the generalised search rule gives

$$S_0^* = 0, S_1^* = 1, n_1^* \geq 1 \quad 6-6-1$$

for Stigleresque search and

$$S_0^* = 0, \text{ and } n_i^* = 1 \text{ for all } i \geq 1 \text{ such that } S_1^* = \dots = S_i^* = 0$$

6-6-2

for sequential search.

If one of these search behaviours is optimal then it must maximise the searcher's present valued ex ante expectations of the net utility attainable from search, $V_0(\xi^*, v^*, \delta^*)$. An examination of $V_0(\xi^*, v^*, \delta^*)$ will, therefore, reveal the circumstances in which the optimal generalised search behaviour coincides with either of the behaviours specified by (6-6-1) or (6-6-2). For this examination it is convenient to express $V_0(\xi^*, v^*, \delta^*)$ in the form given in Lemma 6-6-1, which is comparable to Lemma 2-5-1. Both Lemmas express the ex ante expected net utility from search as the expectation of the sum of net utilities expected ex ante at the ends of periods T_1, T_2, \dots weighted by their ex ante probabilities of being the payoff from search.

Recall, from (1-4-21), the vector $\Psi = (\Psi_j)$ where $\Psi_j(y_j) = \psi_j = 1$ if and only if search is halted at the end of period T_j and where $\Psi_j(y_j) = \psi_j = 0$ otherwise. The values ψ_1, \dots depend upon the search rule $\rho = (\xi, v, \delta)$ chosen since ψ_j may be expressed as

$$\psi_j = (1-S_0)(1-S_1) \dots (1-S_{j-1})S_j \quad 6-6-3$$

for all $j \geq 0$. The intuitive argument given in Section 2-5 for the use of the vector Ψ in the result of Lemma 2-5-1 is directly applicable to the use of Ψ in Lemma 6-6-1.

Lemma 6-6-1:

$$V_0(\xi^*, v^*, \delta^*) = \psi_0^* I(\delta^*(y_0)) + \sum_{j=1}^{\infty} \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_j^* \{ (1-\alpha)^j I(\delta^*(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, n_v^*) \} f_g(y_j) dy_j$$

Proof:

By analogy to (6-2-27),

$$\begin{aligned}
 V_0(\xi^*, v^*, \delta^*) = & S_0^* I(\delta^*(y_0)) + (1-S_0^*) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_1^* \{ (1-\alpha) I(\delta^*(y_1)) - \\
 & (1-\alpha) K(1, n_1^*) \} + (1-S_1^*) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_2^* \{ (1-\alpha)^2 I(\delta^*(y_2)) - \\
 & \sum_{v=1}^2 (1-\alpha)^v K(v, n_v^*) \} + \dots + (1-S_{j-1}^*) \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} [S_j^* \{ (1-\alpha)^j \\
 & I(\delta^*(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, n_v^*) \} + \dots] f_g(y_j | y_{j-1}) dy_j \dots] \\
 & f_g(y_2 | y_1) dy_2] f_g(y_1) dy_1
 \end{aligned} \tag{6-6-4}$$

$$\begin{aligned}
 = & S_0^* I(\delta^*(y_0)) + \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (1-S_0^*) S_1^* \{ (1-\alpha) I(\delta^*(y_1)) - \\
 & (1-\alpha) K(1, n_1^*) \} f_g(y_1) dy_1 + \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (1-S_0^*) (1-S_1^*) S_2^* \{ (1-\alpha)^2 I(\delta^*(y_2)) - \\
 & \sum_{v=1}^2 (1-\alpha)^v K(v, n_v^*) \} f_g(y_2) dy_2
 \end{aligned}$$

+

6-6-5

$$\begin{aligned}
 + & \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} (1-S_0^*) \dots (1-S_{j-1}^*) S_j^* \{ (1-\alpha)^j I(\delta^*(y_j)) - \\
 & \sum_{v=1}^j (1-\alpha)^v K(v, n_v^*) \} f_g(y_j) dy_j
 \end{aligned}$$

+ ...

$$= \psi_0^* I(\delta^*(y_0)) + \sum_{j=1}^{\infty} \int_{p_1^L}^{p_1^U} \dots \int_{p_1^L}^{p_1^U} \psi_j^* \{(1-\alpha)^j I(\delta^*(y_j)) -$$

$$\sum_{v=1}^j (1-\alpha)^v K(v, n_v^*) \} f_g(y_j) dy_j \quad 6-6-6$$

by (6-6-3).

Q.E.D.

The result of Lemma 6-6-1 is that

$$V_0(\xi^*, v^*, \delta^*) = \psi_0^* I(\delta^*(y_0)) + \sum_{j=1}^{\infty} E[\psi_j^* \{(1-\alpha)^j I(\delta^*(y_j)) -$$

$$\sum_{v=1}^j (1-\alpha)^v K(v, n_v^*) \}] \quad 6-6-7$$

Suppose Stigleresque search is optimal. By (6-6-1), $S_0^* = 0$ and $S_1^* = 1$ so, by (6-6-3), $\psi_0^* = 0$ and $\psi_1^* = 1$. Therefore, from (6-6-7),

$$V_0(\xi^*, v^*, \delta^*) = E[(1-\alpha)I(\delta^*(y_1)) - (1-\alpha)K(1, n_1^*)] \quad 6-6-8$$

$$\text{ie. } I(\delta^*(y_0)) < (1-\alpha)E[I(\delta^*(y_1))] - (1-\alpha)K(1, n_1^*) \quad 6-6-9$$

$$\text{and } I(\delta^*(y_1)) \geq \sum_{j=2}^{\infty} E[\psi_j^* \{(1-\alpha)^{j-1} I(\delta^*(y_j)) - \sum_{v=2}^j (1-\alpha)^{v-1} K(v, n_v^*) \} | y_1]$$

6-6-10

for any vector y_1 of observed prices. Under what conditions will (6-6-9) and (6-6-10) be true? (6-6-10) means that it is never optimal to extend a search past the end of period T_1 . This could be caused by

(i) psychic search costs for periods T_2 onwards being prohibitively large and/or

(ii) financial search costs in these periods being prohibitively large.

(iii) the discount rate α being sufficiently large, (6-6-9) will be true only if the transaction cost t and the discount rate α are both within some bounds which will depend upon the intensity of the searcher's preference for commodity 1. There will be many costs and discounting situations where (6-6-9) and (6-6-10) will be true and, for each of them, the optimal search behaviour specified by ρ^* will be Stigleresque.

Now suppose that sequential search is optimal. Then $S_0^* = 0$, which implies $\psi_0^* = 0$, and $n_i^* = 1$ for all $i \leq j$ where $\psi_j^* = 1$. Hence, from (6-6-7),

$$V_0(\xi^*, v^*, \delta^*) = \sum_{j=1}^{\infty} E[\psi_j^* \{ (1-\alpha)^j I(\delta^*(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, 1) \}] \quad 6-6-11$$

(6-6-11) implies that

$$I(\delta^*(y_0)) < \sum_{j=1}^{\infty} E[\psi_j^* \{ (1-\alpha)^j I(\delta^*(y_j)) - \sum_{v=1}^j (1-\alpha)^v K(v, 1) \}] \quad 6-6-12$$

$n_i^* = 1$ for all $i \leq j$ where $\psi_j^* = 1$ implies that, for all vectors of observed prices y_i , the searcher expects it to be optimal to make only one observation on p_1 in period T_{i+1} , assuming he wishes to carry his search on through period T_{i+1} ie.

$$\begin{aligned} & \max\{ (1-\alpha)E[I(\delta^*(y_{i+1})) | y_i] - (1-\alpha)K(i+1, n_{i+1}^*), \\ & \quad (1-\alpha)^2 E[V_{i+2}(\xi^*, v^*, \delta^*) | y_i] - (1-\alpha)^2 K(i+2, n_{i+2}^*) - (1-\alpha)K(i+1, n_{i+1}^*) \} \\ & = \max\{ (1-\alpha)E[I(\delta^*(y_{i+1})) | y_i] - (1-\alpha)K(i+1, 1), \quad 6-6-13 \\ & \quad (1-\alpha)^2 E[V_{i+2}(\xi^*, v^*, \delta^*) | y_i] - (1-\alpha)^2 K(i+2, 1) - (1-\alpha)K(i+1, 1) \} \end{aligned}$$

Conditions under which it is never optimal to take more than one observation per period could be (i) in any period T_i , the marginal psychic search costs of observations subsequent to the first observation p_1^{il} are prohibitively large and/or

(ii) in any period T_i , the marginal financial search costs of observations subsequent to the first observation p_1^{il} are prohibitively large.

(iii) costs are never such as to induce the searcher to take no observations in a period and then take observations in a later period. Such a situation could arise if all the marginal costs in a period were large while the marginal costs of first observations in later periods were small.

In similar fashion to (6-6-9), (6-6-12) implies, for a given intensity of searcher preference for commodity 1, some bounds on the values of t and α for which sequential search will be optimal. There will be many costs and discounting situations satisfying (6-6-12) and (6-6-13) and, for each of these, the optimal search behaviour prescribed by ρ^* will be sequential.

In the light of the debate of the respective optimalities of Stigleresque and sequential search behaviours, it is interesting to note that if the conditions of search are such that $S_0^* = 0$, $S_1^* = 1$ and $n_1^* = 1$, then the two search behaviours are indistinguishable.

For either of the Stigleresque or sequential extremes discussed, a relaxation of the restrictions giving rise to them causes search behaviours to emerge which cannot be described by either of them. For instance, if search costs in period T_2 are reduced, a searcher who previously halted his search at the end of period T_1 may be

induced to extend his search through period T_2 . A Stigleresque search rule no longer describes the searcher's generalised search behaviour. Similarly, if the marginal costs of second observations in some periods are reduced, a searcher may be induced to take at least two observations in some periods. A sequential search rule no longer describes the searcher's generalised search behaviour. One can expect, therefore, that as the technology of search rises, with an accompanying relaxation of the constraints binding a searcher, the extremes of Stigleresque and sequential search will merge into their common ground of generalised search.

CHAPTER VII

CONCLUSIONS

The initial controversy which appeared in the search literature was the controversy concerning sequential search rules' "superiority" over fixed-sample-size search rules such as those discussed by Stigler in his seminal papers [50], [51]. The controversy continues today, and is accompanied by assorted empirical efforts to estimate the margin by which sequential search rules are "superior" when compared to fixed-sample-size search rules. These empirical efforts, however, all compare the properties of these two types of rules in models where the assumed conditions of search are such that sequential search is optimal. Inevitably, therefore, the empirical results show sequential search rules to be superior to fixed-sample-size search rules. These "results" are, of course, of no use at all in answering questions about the relative merits of these two types of search rules in any other type of search problem. In any event, the entire controversy is something of a non-question. Chapter 6 of this thesis demonstrated that sequential search rules were optimal for some search problems, that fixed-sample-size search rules were optimal for other search problems, and that neither type of rule is generally optimal. It is clear that search theorists have largely overlooked that there is no one "search problem" and that the search rule which is optimal for a particular search problem is dependent on the nature of that problem. Insufficient attention has been paid to differences in search problems before the results of analyses based on different problems have been compared. Differences in search problems will also complicate efforts to estimate the opportunity costs of using whichever types

of search rules are optimal for these problems, since the opportunity costs will depend upon which particular search problems are being compared.

Chapters 2, 3 and 4 consider different aspects of search problems for which sequential search rules are optimal. Chapter 2 examined some previous authors' analyses of such problems. It was found that their models' use of a restriction of fixed demand for the searched-for commodity and their use of an expenditure minimisation proxy for utility maximisation was valid only under conditions so restrictive that these analyses were devoid of useful economic content. The sequential search model constructed in Chapter 2 offers a much more fruitful description of these problems by using the indirect utility function to quite naturally incorporate financial search costs, and to make fixed demand restrictions unnecessary. This approach also has the valuable merit of making explicit the hitherto ignored important fact that the consumer's allocation problem is an integral part of his decision making during search. Recognition of this allows a much greater harvest of comparative statics results than is possible in expenditure minimising models.

The rational choice of a set of preferred sellers by the searcher has, like his allocation problem, been largely ignored by the search literature. Excluding the notion of the direction of search has denied the search literature the opportunity to contribute to a variety of topics eg. studies of advertising strategies and location theory. To ignore the direction of search is equivalent to imposing particular severe restrictions on the types of search problems considered. Presumably previous authors chose to ignore the question of selecting

a search path in an effort to prevent their analyses from becoming unmanageably complex. It is ironical, therefore, that including the rational choice of the direction of search in the analysis of search problems not only widens the set of search problems to which the analysis is applicable but, in some instances, also simplifies the analysis. For instance, in Section 3-7, consideration of optimal search paths showed that, under some assumptions on prices and search costs, which are much weaker than those made by the previous authors considered, the form of the sequential search rule reduces to its much simpler myopic form. Chapter 3 and Section 5-4 show some differences in the determination of preferred search paths by sequential and Stigleresque searchers.

Search problems were originally formulated in an effort to explain persistent price dispersion. Various results have been inferred, from both theoretical and empirical studies, about the maintenance of price dispersion over time and about changes brought about in measures of price dispersion by changes in parameters affecting individuals' search decisions. Chapter 4 provides the first derivation of a p.d.f. of transaction prices resulting from sequential search, albeit a derivation subject to strong simplifying assumptions. Comparison of the inferred results to comparative statics results derived from this p.d.f. shows that, while the inferred results are true in some circumstances, they are not true in general. For instance, a commonly inferred result is that a decrease in search costs causes an increase in the quantity of sequential search undertaken and a decrease in price dispersion. The comparative statics results show that, while a decrease in search costs cannot cause a decrease

in the quantity of sequential search undertaken, it may cause an increase in price dispersion. In addition, Chapter 4 shows that the measures of price dispersion commonly used cannot adequately describe price dispersion in any but one unusual market situation.

A major objective of analysing search problems is the formulation of a dynamic non-tâtonnement price adjustment model which provides an explanation of the persistence of, and changes in, price dispersion. At present, the major stumbling block to this formulation is that no general description of searcher behaviour is available. This thesis has made a step in this direction by rationalising three different search models. Furthering this rationalisation to provide a general description of searcher behaviour would seem to be an essential initial step in understanding the processes by which markets make disequilibrium adjustments, the rates at which these adjustments occur, and in understanding how to influence both of these.

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REFERENCES

- [1] Axell, B. Price Dispersion and Information - An Adaptive Sequential Search Model, *Swedish Journal of Economics* 76, 1974; 77-98.
- [2] Burdet, K. *Systematic Search, Search Order, and Risk in a Labour Market*. Paper presented to the Third World Congress of the Econometric Society, Toronto, Canada, August 1975.
- [3] Butters, G.R. *Equilibrium Distributions of Sales and Advertising Prices*. Unpublished thesis, Dept. of Economics, University of Chicago.
- [4] Cyert, R.M. and De Groot, M.H. Adaptive Utility Theory. In Day, R.H. and Groves, T.W. (ed.), *Adaptive Economic Models*, North Holland, 1975.
- [5] Cyert, R.M. and De Groot, M.H. Rational Expectations and Bayesian Analysis, *Journal of Political Economy* 82, No. 3, May/June 1974; 521-536.
- [6] Danforth, J.P. Expected Utility, Mandatory Retirement and Job Search, *Discussion Paper No. 74-41*, June 1974, Dept. of Economics, University of Minnesota.
- [7] Danforth, J.P. Expected Utility, Infinite Horizon and Job Search, *Discussion Paper No. 74-42*, June 1974, Dept. of Economics, University of Minnesota.
- [8] Debreu, G. *The Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New York, John Wiley and Sons, 1959.
- [9] De Groot, M.H. *Optimal Statistical Decisions*. New York, McGraw-Hill, 1970.

- [10] Fisher, F.M. Quasi-competitive Price Adjustment by Individual Firms: A Preliminary Paper, *Journal of Economic Theory* 2, 1970: 195-206.
- [11] Gastwirth, J.L. On Probabilistic Models of Consumer Search for Information. *Quarterly Journal of Economics* 90, No. 1, February 1976: 38-50.
- [12] Gorman, W.M. Tastes, Habits and Choices, *International Economic Review* 8, 1967: 218-222.
- [13] Hadar, J. *Mathematical Theory of Economic Behaviour*. Reading Massachusetts, Addison-Wesley, 1971.
- [14] Haque, W. Samuelson's Self-Dual Preferences. *Econometrica* 43, No. 1, January 1975: 31-39.
- [15] Hey, J.D. Price Adjustment in an Atomistic Market. *Journal of Economic Theory* 8, 1974: 483-499.
- [16] Hotelling, H. Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions. *Journal of Political Economy* 40, No. 5, October 1932: 577-616.
- [17] Houthakker, H.S. Compensated Changes in Quantities and Qualities Consumed, *Review of Economic Studies* 19, 1952: 155-164.
- [18] Ioannides, Y.M. Market Allocation Through Search: Equilibrium Adjustment and Price Dispersion. *Journal of Economic Theory* 11, 1975: 247-262.
- [19] Kihlstrom, R. A General Theory of Demand for Information About Product Quality. *Journal of Economic Theory* 8, 1974: 413-439.
- [20] Kohn, M.G. and Shavell, S. The Theory of Search. *Journal of Economic Theory* 9, 1974: 93-123.

- [21] Krelle, W. Dynamics of the Utility Function. In Hicks, J.R. and Weber, W. (eds.). *Carl Menger and the Austrian School of Economics*. Oxford, Clarendon Press, 1973.
- [22] Lancaster, K. *Mathematical Economics*. MacMillan, New York, 1968.
- [23] Lippman, S.A. and McCall, J.J. Job Search in a Dynamic Economy. *Journal of Economic Theory* 12, 1976: 365-390.
- [24] Lippman, S.A. and McCall, J.J. The Economics of Job Search: A Survey, Part 1. *Economic Enquiry* 14, June 1976: 155-189.
- [25] Lippman, S.A. and McCall, J.J. The Economics of Job Search: A Survey, Part II. *Economic Enquiry* 14, Sept. 1976: 347-368.
- [26] Long, N.V. and Manning, R. *Adaptive Demand Theory: The Case with a Linear Utility Function*. Mimeographed, University of New South Wales, 1976.
- [27] Lucas, R.E. and Prescott, E.C. Equilibrium Search and Unemployment. *Journal of Economic Theory* 7, 1974: 188-209.
- [28] Manning, R. Information and Seller's Behaviour. *Australian Economic Papers* 15, 1976: 308-321.
- [29] Manning, R. and Morgan, P.B. *An Indirect Utility Function Approach to the Search Problem*. Paper presented to the Fifth Conference of Economists, Brisbane, Australia, August 1975.
- [30] McCall, J.J. Probabilistic Microeconomics. *Bell Journal of Economics and Management Science* 2, No. 2, 1971: 403-433.
- [31] McCall, J.J. The Economics of Information and Optimal Stopping Rules. *Journal of Business* 38, No. 3, July 1965: 300-317.
- [32] McCall, J.J. Economics of Information and Job Search. *Quarterly Journal of Economics* 84, No. 1, February 1970: 113-126.

- [33] Mortensen, D.T. A Theory of Wage and Employment Dynamics,
In Phelps, E.S. and others. *Microeconomic Foundations of
Employment and Inflation*, New York, Norton, 1970.
- [34] Nelson, P. Advertising as Information, *Journal of Political
Economy* 82, No. 4, July/August 1974: 729-754.
- [35] Nelson, P. Information and Consumer Behaviour, *Journal of
Political Economy* 78, No. 2, March/April 1970: 311-329.
- [36] Olsen, E.O. A Competitive Theory of the Housing Market.
American Economic Review 59, 1969: 612-622.
- [37] Pissarides, C.A. Job Search and Participation. *Economica* 43,
No. 169, February 1976: 33-49.
- [38] Pollak, R.A. Conditional Demand Functions and the Implications
of Separable Utility. *Southern Economic Journal* 37, No. 4,
April 1971: 423-433.
- [39] Pollak, R.A. Generalised Separability. *Econometrica* 40, No. 3,
May 1972: 431-453.
- [40] Richter, M. Revealed Preference Theory, *Econometrica* 34, 1966:
635-645.
- [41] Rothschild, M. Models of Market Organisation with Imperfect
Information: A Survey. *Journal of Political Economy* 81, No. 6,
November/December 1973: 1283-1308.
- [42] Rothschild, M. A Two-Armed Bandit Theory of Market Pricing.
Journal of Economic Theory 9, 1974: 185-202.
- [43] Rothschild, M. Searching for the Lowest Price when the Distrib-
ution of Prices is Unknown. *Journal of Political Economy* 82,
No. 4, July/August 1974: 689-711.
- [44] Roy, R. *De L'Utilité: Contribution à la Théorie des Choix*.
Paris, 1943.

- [45] Roy, R. La Distribution du Revenu entre les divers Biens. *Econometrica* 15, 1947: 205-225.
- [46] Salop, S.C. Systematic Job Search and Unemployment. *Review of Economic Studies* 40, 1973: 191-201.
- [47] Salop, S.C. and Stiglitz, J. Bargains and Ripoffs: A Model of Monopolistically Competitive Price Dispersions. *Technical Report No. 198*, March 1976, Institute for Mathematical Studies in the Social Sciences, Stanford University.
- [48] Samuelson, P.A. Using Full Duality to Show that Simultaneously Additive Direct and Indirect Utilities Implies Unitary Price Elasticity of Demand. *Econometrica* 33, No. 4, October 1965: 781-796.
- [49] Sato, R. Self-Dual Preferences. *Econometrica* 44, No. 5, September 1976: 1017-1032.
- [50] Stigler, G.J. The Economics of Information. *Journal of Political Economy* 69, No. 3, June 1961: 213-225.
- [51] Stigler, G.J. Information in the Labour Market. *Journal of Political Economy* 70, No. 5, suppl., October 1962: 94-105.
- [52] Telser, L.G. Searching for the Lowest Price. *American Economic Review* 63, No. 2, May 1973: 40-49.
- [53] Telser, L.G. *Equilibrium Price Distributions*. Unpublished paper, July 1974, Dept. of Economics, University of Chicago.
- [54] Veblen, T. Why is Economics not an Evolutionary Science? *Quarterly Journal of Economics* 12, 1898: 373-397.
- [55] Whipple, D. A Generalised Theory of Job Search. *Journal of Political Economy* 81, No. 5, September/October 1973: 1170-1188.
- [56] Yahav, J.A. On Optimal Stopping. *Annals of Mathematical Statistics* 37, No. 1, February 1966: 30-35.

APPENDIX I

A LIST OF NOTATION

This appendix is an alphabetical listing of the notation used in the thesis. English alphabet and script symbols are listed first. Greek alphabet symbols are listed last.

- $A_j = \{a_0, a_{11}, \dots, a_{jn_j}\}$ - the searcher's terminal action space at the end of period T_j .
- a_0 - the terminal action which is to choose to purchase no units of commodity 1.
- a_{ik} - the terminal action which is to choose to purchase a quantity $x_1^* > 0$ of commodity 1 from the k th seller contacted in period T_i .
- c_{ik} - the marginal financial cost to the searcher of contacting the k th seller in period T_i .
- c_i - the marginal financial cost to a sequential searcher of his i th observation. Note that $c_i \equiv c_{i1}$.
- $c(j)$ - the cumulative financial cost to the searcher of taking $\sum_{i=1}^j n_i$ observations on p_1 through periods T_1, \dots, T_j .
- $c_{(j)}$ - the j th smallest marginal financial search cost.
- c_{i_j} - the marginal financial cost to the searcher of contacting the j th seller on the search path i_1, \dots, i_j, \dots .
- $c_{i_j^*}$ - the marginal financial cost to the searcher of contacting the j th seller on the optimal search path $i_1^*, \dots, i_j^*, \dots, i_{j^*}^*$.
- $c_1'(j), c_1''(j)$ - the first and second derivatives, w.r.t. j , of a Stigleresque searcher's financial cost function, under the assumptions that j is continuous and $c(j)$ is twice differentiable w.r.t. j .

$C_j(c_{j+1}, p_1^1, \dots, p_1^j)$ - the searcher's j th reservation price function.

$C_j = C_j(c_{j+1}, p_1^1, \dots, p_1^j)$ - the value of the searcher's j th reservation price.

$C, V,$ - the coefficient of variation for the p.d.f. of the minimum price observed by a sequential searcher.

$E_f[p_1|w]$ - the mean selling price for commodity 1.

$E_h[p_1^{\min}|w]$ - the mean of the p.d.f. of the minimum price observed by a sequential searcher.

$E[j|w_{i_1}, \dots, w_{i_j}, t_{i_1}, \dots, t_{i_j}, c_{i_1}, \dots, c_{i_j}]$ - the ex ante expected sequential search length of a searcher following a search path i_1, \dots, i_j .

$EI(j)$ - the ex ante expected utility, exclusive of psychic search costs, for a Stigleresque search of j observations.

$ETE(i_1, \dots, i_j)$ - the ex ante expected total expenditure on the purchase of a unit of commodity 1, for a sequential searcher following the search path i_1, \dots, i_j .

$f(p_1|w)$ - the p.d.f. of selling prices of commodity 1.

$F(p_1|w)$ - the c.d.f. of selling prices of commodity 1.

$f(p_1|w_{i_j})$ - the p.d.f. of the searcher's beliefs about the probabilities of the possible values of the selling price offered by the j th seller on his search path.

$F(p_1|w_{i_j})$ - the c.d.f. of the searcher's beliefs about the probabilities of the possible values of the selling price offered by the j th seller on his search path.

$f_g(p_1^{11}, \dots, p_1^{jn_j})$ - the p.d.f. of the searcher's ex ante beliefs about the probabilities of the possible values of the observations he intends to take in periods T_1, \dots, T_j (see Appendix II).

$f_g(p_1^{j+1,1}, \dots, p_1^{j+1,n_{j+1}} | y_j)$ - the p.d.f. of the searcher's beliefs about the probabilities of the possible values of the n_{j+1} observations he intends to take in period T_{j+1} given the values of the observations taken in periods T_1, \dots, T_j (see Appendix II).

$g(w)$ - the searcher's prior p.d.f. on w .

$g(w | y_j)$ - the searcher's posterior p.d.f. on w , given the values of the observations taken in periods T_1, \dots, T_j (see Appendix II).

$h(p_1^{\min} | w)$ - the p.d.f. of the minimum price observed by a searcher.

$H(p_1^{\min} | w)$ - the c.d.f. of the minimum price observed by a searcher.

I^+ - the set of non-negative integers.

$I(p_1, p_2, \dots, p_\ell, \bar{M})$ - the searcher's indirect utility function.

$\tilde{I}_0(\bar{M}_j)$ - the indirect utility attainable with prices p_2, \dots, p_ℓ and wealth \bar{M}_j , when demand for commodity 1 is constrained to be zero.

$\tilde{I}_1(\bar{M}_j - (p_1 + t)_j^{\min})$ - the indirect utility attainable with prices p_2, \dots, p_ℓ , wealth \bar{M}_j , and a lowest combined observed selling price and transaction cost for commodity 1 of $(p_1 + t)_j^{\min}$, when demand for commodity 1 is constrained to be one unit.

$I(\delta(y_j))$ - the indirect utility attainable with a terminal decision procedure δ and a vector of observed prices y_j .

i_1, \dots, i_j - a search path over sellers with indices i_1, i_2 to i_j .

$i_1^*, \dots, i_{j^*}^*$ - an ex ante optimal search path.

K_{ik} - the marginal psychic cost to the searcher of contacting the k th seller in period T_i .

$K_1(j)$ - the total psychic cost of search incurred by a Stigleresque searcher's taking j observations.

- $K(j, n_j)$ - the total psychic cost of search incurred in period T_j by taking n_j observations in period T_j .
- K_{1i_j} - the marginal psychic cost to the searcher of contacting the j th seller on the search path i_1, \dots, i_j, \dots .
- $K_{1i_j^*}$ - the marginal psychic cost to the searcher of contacting the j th seller on the optimal search path $i_1^*, \dots, i_j^*, \dots, i_{j^*}^*$.
- M - the searcher's initial wealth.
- \bar{M}_j - the searcher's wealth, net of financial search costs, at the end of period T_j .
- n_i - the number of observations on p_1 taken in period T_i .
- $\underline{n}_{j+1}^J = (n_{j+1}^J, \dots, n_J^J) = v_j^J(y_j)$ - the vector of numbers of observations on p_1 a searcher using an observation number rule v^J intends, at the end of period T_j , to take in periods T_{j+1}, \dots, T_J , when he has observed prices y_j and when there are J search periods.
- $\underline{n}_{j+1}^* = (n_{j+1}^*, n_{j+2}^*, \dots) = v_j^*(y_j)$ - the vector of numbers of observations on p_1 a searcher using an optimal observation number rule v^* intends, at the end of period T_j , to take in periods T_{j+1}, T_{j+2}, \dots , when he has observed prices y_j .
- p_2, \dots, p_ℓ - the prices of commodities 2 to ℓ .
- p_1^L - a lower bound on p_1 .
- p_1^U - an upper bound on p_1 .
- p_1^{ik} - the value of a searcher's k th observation on p_1 in period T_i .
- p_1^i - the value of a sequential searcher's i th observation on p_1 .
Note that $p_1^i \equiv p_1^{i1}$.
- p_{1j}^{\min} - the minimum of the prices observed in periods T_1, \dots, T_j .
- $(p_1 + t)_j^{\min}$ - the minimum of the combined observed selling prices and transaction costs observed in periods T_1, \dots, T_j .

- p_i^* - the fixed point solution to the reservation price equation

$$p_i = C_i(c_{i+1}, p_1^*, \dots, p_{i-1}^*, p_i).$$
- p_{1j}^* - the smallest price for commodity 1 for which a searcher with wealth \bar{M}_j will demand none of commodity 1.
- p_{1j}^{**} - the price for commodity 1 for which a searcher with wealth \bar{M}_j is indifferent between buying none of commodity 1 and demanding a quantity of $x_1(p_{1j}^{**}, p_2, \dots, p_\ell, \bar{M}_j - t)$ from a seller with transaction cost t .
- $\Pr(\psi_k^*=1 | w_{i_1}, \dots, w_{i_j}, c_{i_1}, \dots, c_{i_j}, t_{i_1}, \dots, t_{i_j})$ - the ex ante probability that a sequential searcher takes exactly k observations with an ex ante search path of i_1, \dots, i_j .
- $Q(j)$ - the ex ante expected net utility from a Stigleresque search of j observations.
- $S_j = \xi_j(y_j)$ - the value of the j th element of a stopping rule ξ when prices y_j have been observed.
- $S_j^{bJ} = \xi_j^{bJ}(y_j)$ - the value of the j th element of the backward induction stopping rule ξ^{bJ} when at most J observations will be taken on p_1 and when prices y_j have been observed.
- $S_j^* = \xi_j^*(y_j)$ - the value of the j th element of an optimal stopping rule ξ^* when prices y_j have been observed.
- T_1, \dots, T_j, \dots - a sequence of time periods.
- t_{ik} - the transaction cost for the k th seller contacted in period T_i .
- t_i - the transaction cost for the i th seller contacted by a sequential searcher. Note that $t_i \equiv t_{i1}$.
- $t_{(j)}$ - the j th smallest transaction cost.
- t_{i_j} - the transaction cost for the j th seller on the search path i_1, \dots, i_j, \dots
- $t_{i_j}^*$ - the transaction cost for the j th seller on the optimal search path $i_1^*, \dots, i_j^*, \dots, i_{j^*}^*$.

$TE_j(\delta)$ - the sequential searcher's total expenditure on commodity 1 if search stops at the end of period T_j and the searcher's terminal decision procedure is δ .

$U(x_1, \dots, x_\ell)$ - the searcher's direct utility function.

$U_j^*(0)$ - the direct utility a sequential searcher can attain by taking terminal action a_0 at the end of period T_j .

$U_{ij}^*(1)$ - the direct utility a sequential searcher can attain by taking terminal action a_{i1} at the end of period T_j .

$U_j^*(1)$ - the greatest direct utility a sequential searcher can attain by taking any terminal action other than a_0 at the end of period T_j .

$U(\delta(y_j))$ - the direct utility attained at the end of period T_j if a searcher uses a terminal decision procedure δ to choose a terminal action from his terminal action space A_j .

$\text{var}_h(p_1^{\min} | w)$ - the variance of the p.d.f. of the minimum price observed by a sequential searcher.

$V_j(\xi, \delta)$ - the utility expected, at the end of period T_j , from sequential search conducted according to a sequential search rule $\rho_s = (\xi, \delta)$.

$V_0(\xi, \delta)$ - the utility expected ex ante from sequential search conducted according to a sequential search rule $\rho_s = (\xi, \delta)$.

$V_j^J(\xi, \delta)$ - the utility expected, at the end of period T_j , from sequential search conducted according to a sequential search rule $\rho_s = (\xi, \delta)$, when there are at most J periods in which search can occur.

$V_j(\xi, v, \delta)$ - the present valued net utility expected, at the end of period T_j , from generalised search conducted according to a generalised search rule $\rho = (\xi, v, \delta)$.

- $V_0(\xi, v, \delta)$ - the present valued ex ante expected net utility from generalised search conducted according to a generalised search rule $\rho = (\xi, v, \delta)$.
- $V_j^J(\xi, v, \delta)$ - the present valued net utility expected, at the end of period T_j , from generalised search conducted according to a generalised search rule $\rho = (\xi, v, \delta)$, when there are at most J periods in which search can occur.
- $W_j(\xi, \delta)$ - the total expenditure on the purchase of one unit of commodity 1 expected, at the end of period T_j , from sequential search conducted according to a sequential search rule $\rho_s = (\xi, \delta)$.
- $W_0(\xi, \delta)$ - the ex ante expected total expenditure on the purchase of one unit of commodity 1 from sequential search conducted according to a sequential search rule $\rho_s = (\xi, \delta)$.
- w - a vector of parameters conditioning the p.d.f. of selling prices of commodity 1, $f(p_1|w)$.
- w_{i_j} - the value of w ascribed by the searcher to the j th seller on the searcher's search path i_1, \dots, i_j, \dots .
- $w_{i_j^*}$ - the value of w ascribed by the searcher to the j th seller on the searcher's optimal search path $i_1^*, \dots, i_j^*, \dots, i_{j^*}^*$.
- $w_{(j)}$ - the j th smallest of the values of w ascribed to sellers by the searcher.
- $\bar{w}_{(j)}$ - the j th largest of the values of w ascribed to sellers by the searcher.
- x_i^* - the searcher's demand for commodity i , $i = 1, \dots, \ell$.
- y_j - the vector of the prices observed by the searcher in periods T_1, \dots, T_j .
- α - the discount rate.

γ - an exogenous parameter conditioning psychic search costs.

δ - a terminal decision procedure.

δ^* - an optimal terminal decision procedure.

$v = (v_0, v_1, \dots)$ - an observation number rule.

$v^* = (v_0^*, v_1^*, \dots)$ - an optimal observation number rule.

$v^J = (v_0^J, \dots, v_{J-1}^J)$ - an observation number rule when there are at most J periods in which search can occur.

v^{J*} - an optimal observation number rule when there are at most J periods in which search can occur.

v^{bJ} - the backward induction observation number rule when there are at most J periods in which search can occur.

$\underline{v}_j^J = (v_j^J, \dots, v_{J-1}^J)$ - a truncation of the observation number rule v^J .

$\xi = (\xi_0, \xi_1, \dots, \xi_j, \dots)$ - a stopping rule.

$\xi^J = (\xi_0^J, \xi_1^J, \dots, \xi_J^J)$ - a stopping rule when there are only J search periods.

$\xi^{bJ} = (\xi_0^{bJ}, \xi_1^{bJ}, \dots, \xi_J^{bJ})$ - a backward induction stopping rule when there are only J search periods.

$\xi^{J*} = (\xi_0^{J*}, \xi_1^{J*}, \dots, \xi_J^{J*})$ - an optimal stopping rule when there are only J search periods.

$\xi^* = (\xi_0^*, \xi_1^*, \dots, \xi_j^*, \dots)$ - an optimal stopping rule.

$\rho = (\xi, v, \delta)$ - a generalised search rule.

$\rho^J = (\xi^J, v^J, \delta)$ - a generalised search rule when there are at most J periods in which search can occur.

$\rho^{J*} = (\xi^{J*}, v^{J*}, \delta^*)$ - an optimal generalised search rule when there are at most J periods in which search can occur.

$\rho^* = (\xi^*, v^*, \delta^*)$ - an optimal generalised search rule.

$\rho_s = (\xi, \delta)$ - a sequential search rule.

$\rho_s^* = (\xi^*, \delta^*)$ - an optimal sequential search rule.

$\hat{\rho}_s = (\hat{\xi}, \hat{\delta})$ - a sequential search rule which minimises the ex ante expectation of the total expenditure on the purchase of a unit of commodity 1.

\emptyset - the empty set.

$\phi_i(c_2, \dots, c_{i+1}) = p_i^*$ - the function expressing the i th fixed point solution p_i^* in terms of marginal financial costs of search.

$\Psi = (\Psi_0, \Psi_1, \dots, \Psi_j, \dots)$ - a vector with components Ψ_j which have the property that $\Psi_j(y_j) = \psi_j = 1$ if and only if search halts in period T_j . $\Psi_j(y_j) = \psi_j = 0$ otherwise.

$\psi_j = \Psi_j(y_j)$ - the value of Ψ_j for a given vector y_j of price observations.

$\omega = (\omega_1, \dots, \omega_\ell)$ - the vector of the searcher's initial endowments of commodities 1 to ℓ .

Ω - the set of utility functions for which it is always optimal to search and always optimal to purchase one unit of commodity 1.

APPENDIX II

AN OUTLINE OF BAYESIAN INFERENCE

This appendix gives a short outline of the fundamentals of the statistical tools of Bayesian inference. The reader is referred to De Groot [9, chapters 8 and 12] for a far more complete explanation. The description given here uses the notation of the thesis to assist the reader in understanding how the technique is applied in the thesis.

The source of uncertainty in a Bayesian problem is an unknown parameter w which affects a probability function f over an observation space P_1 . The usual way of denoting this is to write the probability function as $f(p_1|w)$. The Bayesian assumptions, which are informationally demanding, are (i) the Bayesian decision maker has complete knowledge of f except for w .

(ii) the decision maker has a prior probability function g over \mathcal{W} , the space of w .

The decision maker is interested to refine his knowledge of w 's true value because of the higher rewards available to him from making decisions with more accurate information about the true value. Since observations drawn from P_1 are all distributed with probability function $f(p_1|w)$, a sample p_1^1, \dots, p_1^j contains information which the decision maker can use to refine his prior probability function $g(w)$. If values p_1^1, \dots, p_1^j are observed, Bayes theorem gives the probabilities of different values of w as

$$g(w|p_1^1, \dots, p_1^j) = \frac{f(p_1^1, \dots, p_1^j|w)g(w)}{\int_{\mathcal{W}} f(p_1^1, \dots, p_1^j|w)g(w)dw}$$

for all $w \in \mathcal{W}$. This probability function is called the "posterior" probability function of w . The "unconditional" probability function of p_1^1, \dots, p_1^j is the denominator of (A-2-1) and is denoted by

$$f_g(p_1^1, \dots, p_1^j) = \int_{\mathcal{W}} f(p_1^1, \dots, p_1^j | w) g(w) dw \quad \text{A-2-2}$$

As $j \rightarrow \infty$, $g(w | p_1^1, \dots, p_1^j)$ approaches a limiting distribution which is degenerate at the true value of w . The probabilities of values of a $(j+1)$ th observation p_1^{j+1} as assessed by the decision maker after observing p_1^1, \dots, p_1^j are, therefore,

$$f_g(p_1^{j+1} | p_1^1, \dots, p_1^j) = \int_{\mathcal{W}} f(p_1^{j+1} | w) g(w | p_1^1, \dots, p_1^j) dw \quad \text{A-2-3}$$

APPENDIX III

SUPERMARTINGALES AND STOPPING RULES

The purpose of this appendix is to provide the reader with an explanation of how the consumer search problem can take on the structure of a supermartingale. De Groot [9, p. 353-369] gives an excellent and well referenced presentation of the properties of supermartingales and the relationships between supermartingales and optimal sequential stopping rules.

Let $Z_j = z(p_1^1, \dots, p_1^j)$ be a random variable whose value depends on the first j observations p_1^1, \dots, p_1^j .

Definition:

The sequence Z_1, Z_2, \dots is a supermartingale w.r.t. the sequence p_1^1, \dots, p_1^j if, for $j = 1, 2, \dots$, $E[Z_j]$ exists and, with probability 1, $E[Z_{j+1} | p_1^1, \dots, p_1^j] \leq Z_j$. A-3-1

Definition:

If the sequence Z_1, Z_2, \dots is a supermartingale w.r.t. the sequence p_1^1, p_1^2, \dots and if the expected value of Z from search $E[Z_{j*}] \leq E[Z_1]$ for every stopping rule for which $E[Z_{j*}]$ exists, then the supermartingale Z_1, Z_2, \dots is regular.

In the context of a consumer's sequential search problem, the utility attainable after j observations p_1^1, \dots, p_1^j is

$$\max\{U_j^*(0), U_j^*(1)\} = \max\{\tilde{I}_0(\bar{M}_j), \tilde{I}_1(\bar{M}_j - (p_1 + t)_j^{\min})\} \quad \text{A-3-2}$$

The utility attainable is thus dependent on the observations p_1^1, \dots, p_1^j and the variable $\max\{U_j^*(0), U_j^*(1)\}$ is of the form of Z_j . Suppose that, after j observations have been taken, the sequence of the expected utilities from search has the property that

$$E[\max\{U_{j+k}^*(0), U_{j+k}^*(1)\} | p_1^1, \dots, p_1^j] \leq \max\{U_j^*(0), U_j^*(1)\} \quad A-3-3$$

for all $k = 1, 2, \dots$. Comparing (A-3-3) to (A-3-1) shows that such a sequence of expected utilities is a supermartingale w.r.t. future observations $p_1^{j+1}, p_1^{j+2}, \dots$.

Theorem: (see De Groot [9, p. 359])

Let the sequence Z_1, Z_2, \dots be a supermartingale and suppose that the random variables Z_1, Z_2, \dots are uniformly integrable. Then $E[Z_{j*}]$ exists for any stopping rule for which the probability of only a finite number of observations being taken is unity, and $E[Z_{j*}] \leq E[Z_1]$.

The regularity property of the supermartingale sequence of expected utilities (A-3-3) is thus established if, for all $j = 1, 2, \dots$, the terms in (A-3-2) are shown to be uniformly integrable ie. for all $j = 1, 2, \dots$

$$\Pr(|\max\{\tilde{I}_0(\bar{M}_j), \tilde{I}_1(\bar{M}_j - (p_1 + t)^{\min})\}| \leq UB) = 1 \quad A-3-4$$

where $UB < \infty$ is an upper bound on the utility attainable from search. Such an upper bound is $\max\{\tilde{I}_0(M), \tilde{I}_1(M - p_1^L)\} < \infty$. Set

$$UB = \max\{\tilde{I}_0(M), \tilde{I}_1(M - p_1^L)\} \quad A-3-5$$

and (A-3-4) is established. The sequence of variables (A-3-2) with the property (A-3-3) is therefore a regular supermartingale.

The economic meaning of this is as follows. After j observations p_1^1, \dots, p_1^j , the searcher expects

$$(i) \quad E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} | p_1^1, \dots, p_1^j] \leq \max\{U_j^*(0), U_j^*(1)\} \quad A-3-6$$

$$(ii) \quad E[\max\{U_{j*}^*(0), U_{j*}^*(1)\} | p_1^1, \dots, p_1^j] \leq E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} |$$

$$p_1^1, \dots, p_1^j] \quad A-3-7$$

where $j^* \geq j+1$. (i) says that the searcher expects to do no better by taking a $(j+1)$ th observation p_1^{j+1} . (ii) says that the searcher does

not expect the utility attainable from a search of $(j+1)$ or more observations to exceed the utility expected to be attainable if only the $(j+1)$ th observation p_1^{j+1} is taken. Overall, therefore, the searcher expects to do no better at any point in his search that is subsequent to his present position of having taken observations p_1^1, \dots, p_1^j . It is rational for him to stop search. This is the basis of the following theorem.

Theorem: (see De Groot [9, p. 367])

Consider a problem of optimal sequential stopping in which an optimal stopping rule exists. Suppose that for any set of observed values p_1^1, \dots, p_1^j for which $E[Z_{j+1} | p_1^1, \dots, p_1^j] \leq Z_j$, the sequence of future values Z_{j+1}, Z_{j+2}, \dots is a regular supermartingale w.r.t. the sequence of future observations $p_1^{j+1}, p_1^{j+2}, \dots$. Then an optimal procedure after any set of values p_1^1, \dots, p_1^j has been observed is to continue sampling if $E[Z_{j+1} | p_1^1, \dots, p_1^j] > Z_j$ and to terminate the sampling process if $E[Z_{j+1} | p_1^1, \dots, p_1^j] \leq Z_j$.

Note that the *only* expectation considered by the searcher is the expectation after *one* additional observation. The theorem states that a myopic sequential search rule is optimal. When the theorem is placed in the context of the consumer search problem, the condition for taking the $(j+1)$ th observation on p_1 is that

$$E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} | p_1^1, \dots, p_1^j] > \max\{U_j^*(0), U_j^*(1)\} \quad \text{A-3-8}$$

which, by (2-5-19), implies that

$$E[V_{j+1}(\xi^*, \delta^*) | y_j] = E[\max\{U_{j+1}^*(0), U_{j+1}^*(1)\} | p_1^1, \dots, p_1^j] \quad \text{A-3-9}$$

(A-3-9) is identical to the condition (2-7-40) which was claimed in Chapter 2 to be sufficient for myopic search to be optimal.

APPENDIX IV

THE PROBABILITY DENSITY FUNCTION OF SAMPLE MINIMA

Sample minima are considered throughout a large portion of this thesis. This appendix gives a derivation of the p.d.f. of the minimum of a sample of j independent observations on p_1 drawn from a large population with p.d.f. $f(p_1|w)$.

$$p_{1j}^{\min} = \min\{p_1^1, \dots, p_1^j\} \quad A-4-1$$

$$\text{Therefore } p_1^i \geq p_{1j}^{\min} \text{ for all } i = 1, \dots, j, \quad A-4-2$$

with equality for at least one of $i = 1, \dots, j$. Consider the probability that $p_{1j}^{\min} > p_1$. Then, by (A-4-2),

$$\Pr(p_{1j}^{\min} > p_1 | w) = \Pr\left(\bigcap_{i=1}^j \{p_1^i > p_1\} | w\right) \quad A-4-3$$

The observations are independently and identically distributed so

$$\Pr(p_{1j}^{\min} > p_1 | w) = \prod_{i=1}^j \Pr(p_1^i > p_1 | w) \quad A-4-4$$

$$= \prod_{i=1}^j (1 - F(p_1 | w)) \quad A-4-5$$

$$= (1 - F(p_1 | w))^j \quad A-4-6$$

The c.d.f. of p_{1j}^{\min} is therefore

$$\Pr(p_{1j}^{\min} \leq p_1 | w) = 1 - (1 - F(p_1 | w))^j \quad A-4-7$$

The p.d.f. of p_{1j}^{\min} is the differential w.r.t. p_1 of (A-4-7),

$$\Pr(p_{1j}^{\min} = p_1 | w) = j (1 - F(p_1 | w))^{j-1} f(p_1 | w) \quad A-4-8$$